Some Tools of the Trade: Numbers, Quantities, and Units

The language of physics: symbols and formulas

Positive and negative numbers
Zero
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Right-angled triangles

Once more the four forces, this time quantitatively

The gravitational force
The electric force
The other forces

So far we have used words, almost exclusively. We have talked about size, force, and energy, but mostly without using numbers, although a few times they almost forced themselves on us. But most of the time we have to know how big a distance is, or a force, or any other quantity, and that requires numbers and units. We also want to describe relations between different quantities, and do that with symbols, such as E for energy, and M for mass.

All of that is *mathematics*. Mathematics is the language in which the ideas, facts, and relationships of physics are best expressed. Sometimes it's just shorthand. It is much easier to write $v = 32$ m/s than "the velocity is thirty-two meters per second," or $\bar{v} = \frac{\Delta x}{\Delta t}$, rather than "the average velocity of an object is equal to the displacement along the x-axis divided by the time it took to make that displacement."

Sometimes the relationships are more complex, like $F = Ma$: "the force, or, if there is more than one, the sum of all the forces acting on an object, is equal to the mass of the object multiplied by its acceleration."

You can see that just as a way of writing things down in shorthand notation with symbols like v and x, math is very helpful. But it does more than that. It lets us write down relationships between different quantities, and change them so that they lead to other relationships. To do the same with words would be cumbersome even for the simplest ones, and close to impossible for others.

Mathematicians can make up their own rules. All that is required is that they are not in conflict with one another, i.e., that they are *internally consistent.* Physics, and more generally science, does something quite different. It describes the world as it exists. We can make up rules and laws, but more than consistency is required of them. The more severe test is whether they help us in the primary task, the description of the dynamic, interacting, ever-changing, endlessly new world that we observe.

2.1 The language of physics: symbols and formulas

Look at the equation $y = 3x^2$. It could be seen as a "formula," or "recipe" that expresses the quantity y in terms of the quantity x: when $x = 1, y = 3$, when $x = 2, y = 12$, and so on. Given x we can follow the rules and find y. A physicist is more likely to see it as a *relationship*, for which the mathematical equation provides the description. The equation shows how y depends on x. In other words, it shows y as a *function* of x. It shows that y is proportional to x^2, and brings to mind a picture of the graph of y against x that shows how the two quantities are related.

Some symbols stand for quantities, like distance, L, and their units, like meters, m. Others describe a procedure or operation, like *plus* $(+)$ or *times* $(\times)$, or a relation like *equals* $(=)$ or "is proportional to" $(\propto)$.

A page with unfamiliar symbols does not give us understanding or comprehension. It is like a page of musical notes that comes alive to the musician as he or she looks at it, but remains hidden to those who do not know musical notation. Moreover, to know what each written note means is far from knowing what an orchestral passage sounds like. It is the same with a page of physics filled with mathematical notation. Our intention is to help you see, follow, and "hear" what is being described.

When we speak or write we use the "parts of speech," the nouns, verbs, adjectives, and so on. Their proper use leads to understanding and communication, while their improper use can lead to confusion. Similarly, to "speak physics" we need mathematical components, such as numbers (positive, negative, and zero), graphs, proportionalities, and equations. In the following sections we review some of the mathematical procedures and ideas that we will use to communicate.

Positive and negative numbers

In physics we need to quantify. We ask *How much? How far? How large?* The answers are expressed in numbers. In many cases we want to distinguish between opposites such as *right and left, past and future, speeding up and slowing down*. We do that by using positive and negative numbers.

Here is an example. I walk six steps to the right and then four steps to the left. How far am I from where I started?

We let the starting point be represented by zero, "to the right" by positive numbers, and "to the left" by negative numbers. We can then describe the motion mathematically as $(+6 \text{ steps}) + (-4 \text{ steps}) = (+2 \text{ steps})$. We use *plus* and *minus* signs, but we use them in two entirely different ways. One (within the parentheses) is to indicate which way is to the right and which to the left. The other (between the parentheses) is to indicate the *operation* of addition.

To distinguish between the two uses we can reserve "plus" and "minus" for the operations of addition and subtraction, and use "positive" and "negative" to indicate positive and negative numbers. Our relation then reads "positive six steps plus negative four steps equals positive two steps." "Positive" and "negative" tell us what kind of numbers we have, and "plus" and "minus" tell us what we do with them.

The choice of which direction to call positive is up to us. If we decide to let "to the left" be positive, we would write $(-6 \text{ steps}) + (+4 \text{ steps}) = (-2 \text{ steps})$. Both choices lead to the same result, namely that we end up two steps to the right.

Sometimes we don't care whether a number is positive or negative; we just want to know how big it is. That's its *magnitude*. The magnitude of both $+15$ (positive 15) and -15 (negative 15) is 15. When we look at the speedometer in a car we see how fast the car is going. That tells us its

speed: the speed is 30 miles per hour. To incorporate the direction, we need the *velocity*: the velocity is 30 miles per hour, north. The speed is the magnitude of the velocity.

EXAMPLE 1

3 units ← | A | → 2 units   3 units ← | B | → 5 units
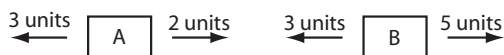
Zahra and Mona pull in opposite directions on a cart (*A*) that is initially at rest. Laila and Yasmine also pull in opposite directions on a second identical cart (*B*). The figure shows the forces that they apply in force units.

(a) Represent these actions by mathematical statements, using numbers.

(b) In which direction does each cart start to move?

(c) Which cart is being pulled with the greatest net force?

Ans.:

(a) Let the positive direction be to the left.
    Cart *A*: $(+3 \text{ units}) + (-2 \text{ units}) = (+1 \text{ unit})$.
    Cart *B*: $(+3 \text{ units}) + (-5 \text{ units}) = (-2 \text{ units})$.

(b) Cart *A* starts to move in the positive direction, to the left, as a result of the net pull (or force) to the left of one unit.

    Cart *B* starts to move in the negative direction, to the right, as a result of the net pull (or force) in that direction of 2 units.

(c) The size or amount of the net force on cart *A* is 1 unit, the size of the net force on cart *B* is 2 units. Cart *B* is being pulled with twice the net force of cart *A*. The negative sign tells us the direction and the number tells us how hard the pull is.

Zero

Zero can have several meanings. It can mean the absence of a quantity, as in "the car's speed is zero." Here the car has no speed at all. It is not moving.

A second meaning is for zero to denote a starting point. From this point we can move in one direction or the other, or the temperature can change up or down. More generally, zero can be a position, or a temperature or the value of some other quantity, from which we count positions, temperatures, etc. This is what we do when we use the *Celsius* scale of temperature, where *zero degrees* refers to the temperature at which water freezes.

A third use of *zero* is to denote a balance or neutral condition between opposites. Two equal forces in opposite directions produce a *zero net force*. A neutral atom has just as many protons (with their positive charge) as electrons (with their negative charge). Its *net electric charge* is zero.

EXAMPLE 2

You go to bed and the temperature is 18°C. Overnight the temperature drops by 18°C.

Write a mathematical expression for what happens, first in the C-scale and then in the F-scale. What is the temperature in the morning and in the evening in degrees Celsius and Fahrenheit?

Ans.:
$t = (18°C) + (-18°C) = 0°C$. The final temperature is 0°C.

The size of a degree C is $\frac{9}{5}$ times the size of a degree F. The magnitude of the drop in temperature is therefore $(18)(\frac{9}{5}) = 32.4$ Fahrenheit degrees.

0°C or 32°F is the freezing temperature. This is also the morning temperature. The evening temperature of 18°C is equal to $32°F + 32.4°F = 64.4°F$.

The final (morning) temperature is 0°C. It is also the evening temperature plus the change in temperature, $(64.4°F) + (-32.4°F) = (32°F)$.

EXAMPLE 3

Two teams are engaged in a tug of war. Each team pulls on the rope with 25 units of force. The rope does not move. Write a mathematical description of the forces with which the rope is being pulled.

Ans.:
Let the positive direction be to the right. $(25 \text{ units}) + (-25 \text{ units}) = (0 \text{ units})$.

The rope is being pulled from both sides as each team exerts a force, but the net force on it is zero.

EXAMPLE 4

The atomic number of sodium is 11. In its most common ionized state each atom loses one electron. Represent the charges of the neutral atom and the ion

using a mathematical description of the numbers of electronic charges.

Ans.:

The proton and the electron have charges of the same magnitude, *e*, positive for the proton and negative for the electron. The neutral sodium atom contains 11 protons and 11 electrons. The net amount of charge is $(+11e) + (-11e) = 0$.

When the atom is ionized and loses one electron, the negative charge is $(-10e)$. The amount of charge of the ion is then $(+11e) + (-10e) = (+1e)$.

(The number of protons in a sodium nucleus is 11, regardless of whether the atom is neutral or whether it has gained or lost electrons.)

Numbers, huge and tiny: powers of 10

When we use atomic or astronomical distances we need to use very small and very large numbers. It is helpful to use a notation that avoids long strings of zeros.

We can write the number 100 as 10^2 and the number 1000 as 10^3. The "2" and the "3" are called "exponents." You can think of them as the number of steps that the decimal point is shifted to the right of the "1." This way of looking at the exponent also works in the other direction: 0.01 is written as 10^{-2}, where the negative sign means that we move the decimal point to the left from its position after the "1."

To multiply two numbers written in this power-of-ten notation we need to add the exponents: $100 \times 1000 = 100,000$, can be written as $10^2 \times 10^3 = 10^5$.

Since 0.01 is equal to $\frac{1}{100}$, we see that 10^{-2} is equal to $\frac{1}{10^2}$.

We can extend the notation to other numbers by writing 6.2×10^3 instead of 6200. Again, the exponent gives the number of steps that the decimal point is moved, either to the right if the exponent is positive, or to the left if it is negative.

EXAMPLE 5

The earth is about 150,000,000,000 m from the sun. Pluto is about 6,000,000,000,000 m from the sun.

(a) What is the ratio of the distances?

(b) What is the radial distance from Pluto's orbit to the orbit of the earth?

Ans.:

Sun to earth: 1.5×10^{11} m.

Sun to Pluto: 6.0×10^{12} m.

Ratio: $\frac{6.0 \times 10^{12}}{1.5 \times 10^{12}} = \frac{6.0}{1.5} \times 10^{(12-11)} = 4.0 \times 10^1 = 40$, so that Pluto is 40 times as far from the sun as is the earth.

Difference: $6.0 \times 10^{12} - 1.5 \times 10^{11}$. This is easiest if both exponents are the same: $6.0 \times 10^{12} = 60 \times 10^{11}$ and $60 \times 10^{11} - 1.5 \times 10^{11} = 58.5 \times 10^{11}$, which can be rounded off to 58 or 59×10^{11} m.

Precision: significant figures

You may describe the distance from the edge of a desk to the nearest wall as 5 feet and 4 inches, which is about 5.3 feet. But if someone tells you that 5 feet 4 inches is really 5.33333 feet, with the threes going on forever, you have to ask yourself how many of these threes to write.

It is true that 4 inches equal $\frac{1}{3}$ ft, or 0.333... ft, with an unlimited number of threes, but we are looking at the results of a measurement, not at an abstract number. To write down that third "3" implies that we know that it is not a "2," a "4," or some other number. If we really know, because the distance was measured sufficiently precisely, then this number should be there, because it is *significant*. Otherwise it should be left out.

If we write the distance as 5.3 ft, we imply that it is bigger than 5.2 feet and smaller than 5.4 feet. That's not a bad measurement. It says that we know the distance to about one part in 53. That's about two parts in 100, or 2%. If we write 5.33 ft, the implication is that it is not 5.32 or 5.34, so that we know the distance to about one part in 533, or 0.2%. You have to ask yourself whether you really know the distance that well.

A calculator can give numbers with many figures, and you have to decide how many of them are significant. If the number is the result of a measurement, the number of significant digits depends on the precision of the measurement. It is most often three figures or less. We usually "round" the last digit that we keep, either down, by leaving out the next one, if it is less than 5, or up, by increasing it by 1 if it is 5 or more. 5.333 becomes 5.33 and 5.336 becomes 5.34.

Sometimes we only know the exponent, that is, we may know only that a number is 10^{23}, and

not 10^{22} or 10^{24}. We then say that we know the *order of magnitude*.

There are rules about what happens to the number of significant figures when numbers are combined. For example, if you multiply or divide two numbers, each with three significant figures, the result should also have three significant figures. In general, the least precise number in a combination will determine how precise the result is. On the other hand, if you are not quite sure, it is better to keep an extra significant figure until the end of a calculation, and only then to round it. Rules are useful, but common sense may also be necessary.

For simplicity we will assume that the quantities in our examples and problems are known to three significant figures unless we say something different. In other words, when we say 2 m, it will imply 2.00 m, and when we give a force as 5.3 N, that will imply 5.30 N. The calculations and answers should therefore also be carried out to three significant figures.

EXAMPLE 6

You have a rope that is 52.3 feet long, which you want to cut into three equal parts. How long should each piece be?

Ans.:
You put the numbers in your calculator and get $\frac{52.3}{3} = 17.43333333$ ft. You now look at the appropriate precision, and see that the length of the rope is given to three significant figures. You therefore round the answer to 17.4 ft. You check that this is reasonable by considering the precision of the measurement. 0.1 ft is about an inch, and this seems about right for the precision of the measurement of the length of a 50 ft rope.

Quantities and units

We have already talked about *energy*. You know that energy is what you get from the gasoline that you put in the car and from the electric company when you plug in a light. It is what you get from the food you eat and what you use when you lift this book. Another term that we have used is *force*. It describes our interaction with a wagon when we pull it and with a wall when we push on it. The baseball bat or the tennis racket interacts with a ball by means of forces, and so do you and the earth when you jump, and even when you stand still.

Terms like *force* and *energy* are used in everyday language, but they are given much more precise meanings in science. We need more than a general idea of what we are talking about. We want to be quantitative, and for that we need definitions that provide recipes for measuring the various quantities. Only then can we say how big a force is and how much energy is required for a particular task.

We will talk about the connections between distance, force, and energy that lead to their precise definitions in the next chapters. In the meantime we will continue to use them, since we already have a rough idea of what these quantities are. Let's add one more, the *mass*. It is the property that tells you how hard it is to get an object to change its velocity. From it you can also see how hard the earth pulls on it. That's the *weight* of the object.

To measure something we need units. Inches, meters, and miles are among the units that are used to measure distance, and only when we know how big each is can we make sense of what is meant by 3 inches, or 5 meters, or 10 miles. If we know how many kilometers are equivalent to 1 mile, we can change from describing a distance as 15 miles to saying that it is some number of kilometers.

For energy we need other units, such as the *kilowatt-hour*, the *calorie*, the *joule*, and the *electron-volt*. These are among the units that let us describe and compare the energy needed to lift this book from the floor to the table, to keep a light on for three hours, or to remove an electron from a hydrogen atom. They also let us describe how much energy a person uses in a day, or how much all of us in this country, or on the whole earth, use in a year.

We start by comparing different units for the same kind of quantity, for example distance. It is helpful to have a fundamental unit, and then to express the others in terms of it. Most of the time we will use the units of the "SI System" (from the French *Système Internationale*) as our fundamental units. The SI unit for distance is the *meter*, m, but distance can also be measured in many other units (miles, inches, centimeters, etc.). Other SI units include the *joule,* J, for energy, the *newton,* N, for force, the *kilogram,* kg, for mass, and the *second,* s, for time.

Although we will wait with the exact definitions of energy, force, and mass, we can

anticipate some of what we will get to. The *kilogram* is the fundamental (SI) unit of mass. The *newton* is the SI unit for force. The weight of an object whose mass is 1 kg (the force that the earth exerts on it near the earth's surface) is about 2.2 pounds, or about 9.8 N.

Suppose you want to express a distance of 6.71 m in feet, using the fact that 1 m = 3.28 ft. The easiest way is to multiply the 6.71 m by a fraction that is equal to one, with the meters in the denominator and the same distance, in feet, in the numerator. 6.71 m is the same as $\frac{6.71 \text{ m}}{1}$, so that we can write $(\frac{6.71 \text{ m}}{1})(\frac{3.28 \text{ ft}}{1 \text{ m}})$.) Since units that appear both on the top and on the bottom cancel, the meters cancel, and we are left with (6.71)(3.28) ft, or 22.0 ft.

EXAMPLE 7

How long (in days and years) would it take you to count 100 million one-dollar bills if you counted them at the rate of two per second?

Ans:
$(2\frac{\$}{s})(t) = 10^8$, or $t = 0.5 \times 10^8$ s.

$$(0.5 \times 10^8 \text{ s})\frac{1 \text{ h}}{3600 \text{ s}}\frac{1 \text{ d}}{24 \text{ h}} = \frac{0.5 \times 10^8}{(3600)(24)} \text{ d} = 579 \text{ d}$$

$$= \frac{579}{365} \text{ y} = 1.6 \text{ y}$$

At this rate it will take 1.6 years!

Ratios and proportional reasoning

The ratio of the circumference of a circle to its diameter is π. This is so regardless of the size of the circle. (π is a number with an infinite number of decimal places. To three figures it is 3.14.) We can write $\frac{c}{d} = \pi$, and $c = d\pi$, where c is the circumference and d is the diameter. We can also write "c is proportional to d" ($c \propto d$), which means that c is equal to a constant times d. If d is multiplied by 2 (or some other number) then c is also. This is an example of "proportional reasoning."

Here is another example. The energy of motion (or "kinetic energy," K) of a car is $\frac{1}{2}mv^2$, where m is its mass and v is its speed. What happens to K when v doubles?

When the speed has the value v_1, K has the value K_1, equal to $\frac{1}{2}mv_1^2$. When the speed is v_2,

the energy is $K_2 = \frac{1}{2}mv_2^2$. $\frac{K_2}{K_1} = \frac{\frac{1}{2}mv_2^2}{\frac{1}{2}mv_1^2}$, which is equal to $\frac{v_2^2}{v_1^2}$ or $(\frac{v_2}{v_1})^2$. When the speed is doubled, $\frac{v_2}{v_1} = 2$, and $(\frac{v_2}{v_1})^2 = 4$, so that the energy is then four times as large.

More simply, we can write $K \propto v^2$ (K is proportional to v^2) since the mass does not change. This shows that when v is multiplied by some number (here 2), K is multiplied by the same number squared. Proportional reasoning allows us to say this without knowing the values of m, v, or K. The equation $K = \frac{1}{2}mv^2$ describes the relation between K and v. The proportionality, $K \propto v^2$, is part of that relation. It allows us to see how K changes when v changes.

Suppose that $P = \rho gh$, where ρ and g are constant. We don't need to know anything about ρ and g to see how P and h depend on each other: we see that $P \propto h$. If h is tripled, so is P. Proportional reasoning allows us to predict the effect on P of a change in h, while all other quantities remain unchanged. In this relation P is the pressure in a liquid whose density is ρ (greek *rho*), at a depth h, and g is the weight in *newtons* of a body whose mass is one *kilogram*, but regardless of what the letters represent, as long as ρ and g are constant, $P \propto h$.

EXAMPLE 8

An approximate relation between the force of air resistance, F_a, on a moving car and its speed v is $F_a = kv^2$. k is a constant quantity that depends on the size and shape of the car. The car speeds up from 40 miles per hour to 60 miles per hour. By what factor will the force of air resistance increase?

Ans.:
$F_a \propto v^2$. v increases from the initial speed $v_i = 40$ mph to the final speed $v_f = 60$ mph, while the force of air resistance changes from its initial value F_{ai} to its final value F_{af}. $\frac{F_{af}}{F_{ai}} = \frac{kv_f^2}{kv_i^2} = (\frac{v_f}{v_i})^2 = (\frac{60}{40})^2 = (1.5)^2 = 2.25$. The final value F_{af} is 2.25 times the initial value F_{ai}, i.e., the force of air resistance increases by a factor of 2.25.

Tables, graphs, equations, and functions

The results of an experiment or an observation can be shown in a table of data. The table might

show values of some quantity x, and for each value of x the corresponding value of some other quantity y. We can also represent the data on a graph of y against x. The table and the graph show how y changes when x changes. They show y as a *function* of x, or in shorthand notation they show $y(x)$, pronounced "y of x."

We try to find patterns in the data. For example, the graph might be a straight line, a parabola, or some other shape, which we can then describe by an equation.

EXAMPLE 9
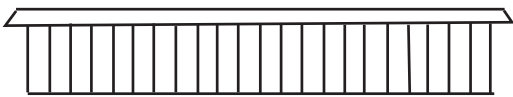
You skate down the sidewalk past a long building with regularly spaced windows. You count the number of windows that you pass. You can then make a table. Let x be the time in seconds and y the number of windows passed.

x time in s	y windows passed
0	0
1	4
2	8
3	12
4	16
5	20

(a)   Make a graph of the data.

(b)   Represent the data by an equation.

(c)   Why is is it physically meaningful to replace the data with a function?

Ans.:
(a)
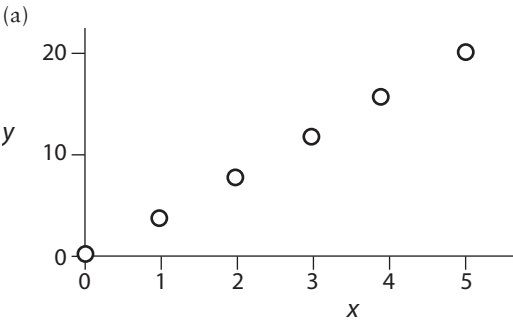
(b and c)   You can draw a line that passes through the points. What do the parts of the line between the points represent? You didn't count anything there, but the line tells you what the measurements would be if the same trend is followed.
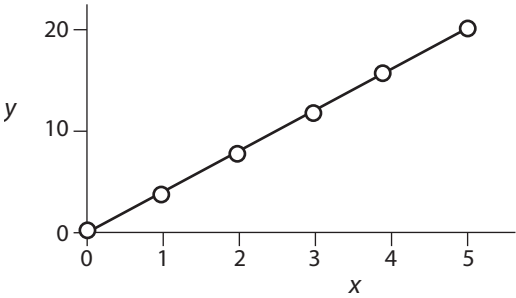
You can express the line on the graph by an equation: the equation that describes the line is $y = 4x$.

Each of the three descriptions (the table, the graph, and the equation) describes the same data. Each gives its particular perspective. Each is a different *representation*. Each illuminates what is happening in a different way.
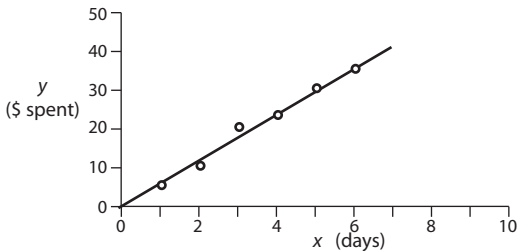
EXAMPLE 10

x (days)	y (\$ spent)
0	0
1	5
2	11
3	20
4	23
5	31
6	35

You start the month with \$100 budgeted for eating out and entertainment. The table shows your expenses as they accumulate day by day for the first week.

(a) Make a graph of the data.

(b) Write the mathematical function that describes the line on the graph.

Ans.:
(a)

The points on the graph lie quite closely, but not exactly, along a straight line. By drawing a straight line through them you assume that the relationship is linear. The line represents your estimate of the "best fit" to the data. It *averages* over the deviations from the straight–line relationship.

(b) The graph shows the amount of money spent (y) plotted against the number of days that have passed (x). It shows y as a *function* of x, or $y(x)$. The equation of the line that we have drawn is $y = 6x$. The slope, the number of dollars spent per day, is 6 $/day. It is positive, showing that y increases as x increases.

Alternatively we can use the same data to plot the amount of money that remains against the number of days:
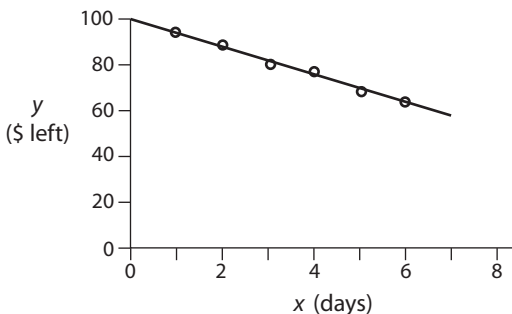
The line starts with $100 at time 0 and decreases by $6 each day, on average. The equation of the line is $y = 100 - 6x$. This time the slope is negative, equal to -6 $/day. The negative sign shows that y is decreasing as x increases.

Both graphs give a description of the same data, but they do so differently. Each is a different representation.

One value of the different representations is that they allow us to find patterns: the skater passes four windows in each second; the money is spent at the constant rate of six dollars per day.

A second value is that they allow us to see what happens at points in between the recorded data. You can look at the graph and see how many windows you have passed after 3.5 seconds. This is called *interpolation*. You can also see how many windows you would pass if you kept going for 10 seconds. This is called *extrapolation*. It gives the correct answer only if the pattern continues unchanged. In this example it will do so only if the windows continue to be equally spaced for a sufficient distance, and if you continue skating in exactly the same way.

For the example in which you spend money at the rate of six dollars a day, you can extrapolate to see when you will run out of money if you continue to spend it at the same rate.

What is the mathematical description of this question? The equation is $y = 100 - 6x$, where y is the money left and x is the number of days. You can then ask what x is when y is zero.

The equation then becomes $0 = 100 - 6x$, and it leads to the solution $6x = 100$, or $x = \frac{100}{6} = 16.7$ days. This shows that with the given rate the amount of money left will go to zero in 16.7 days. You can conclude that unless you change the rate at which you spend money you will not be able to go out to eat or go to a movie for almost half of the month.

Here is another example. A mouse comes out of its hole and moves along the floor in a straight line. That's the event. That's what happens. The words, spoken or on paper, are one representation. The drawing, and a photograph, are other representations. So are graphs of the mouse's position or speed as a function of time. Each representation gives some partial information about what happens.

Here is a table of measurements of the mouse's position. x is the distance from the starting point at the mouse hole, in centimeters. t is the time in seconds, in half-second time intervals, measured from the moment the mouse emerges.

| x (cm)	t (s)
0	0
0.5	26
1	52
1.5	61
2	64
2.5	65
3	63
3.5	59
4	53
4.5	51
5	51
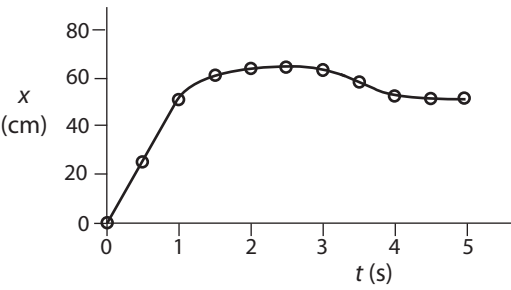
The graph shows the position of the mouse as a function of the time. It shows the *data*, i.e., the results of measurements. To make this graph, a number of decisions had to be made. What quantities do we plot on each axis? What are the units on each axis? What is the scale, i.e., how far apart are the numbers on each axis? Where is each quantity measured from, i.e., where is each of the quantities equal to zero? Is that also the place where the graph has its origin, i.e., where the axes (plural of *axis*) cross? Each of these decisions affects the way the graph looks. Together they describe the *coordinate system*. The facts (the data) are not changed. But the way the facts are represented by a graph is different for each coordinate system.

The graph starts out as a straight line. During this part of its travel the mouse covers equal distances in equal times. In other words its speed is constant: 26 cm in the first half second and 26 cm in the second half second, or 52 cm/s. It then slows down, going only 13 cm (from $x = 52$ cm to $x = 65$ cm in the next 1.5 s.

What happens then? We see that x, the distance from the starting point, decreases during the next two seconds. The mouse is moving back toward the starting point. During the last half second x does not change. The mouse is not moving.
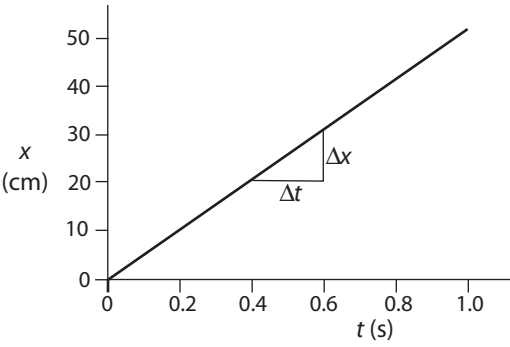
Now zoom in on the part of the graph on the left, near the origin. There the points lie along a straight line. The small triangle represents a change along the x-axis that we call Δx (capital Greek *delta*) and a change along the t-axis that we call Δt. Here Δx is 10.4 cm, and Δt is 0.2 s.

The ratio $\frac{\Delta x}{\Delta t}$ is the *slope* of the line. Here it is $\frac{10.4}{0.2}$ cm/s, or 52 cm/s. If the graph describes physical quantities (not just numbers), each axis must include *units*. Here they are cm and s. The slope has units of cm/s. For this line the slope is 52 cm/s. We see that the slope of the graph of $x(t)$ is the velocity.

On the graph of $y(x)$, y is a function of x. Just numbers are given this time, no units. The graph
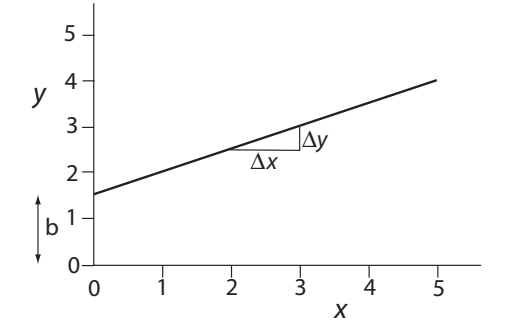
is a straight line. The place where the line crosses the y-axis is called the y-intercept. The symbol used to represent the y-intercept is b. Here it is 1.5. Δy is a change parallel to the y-axis (here 0.5), and Δx is a change parallel to the x-axis (here 1.0). The slope of the line is $\frac{\Delta y}{\Delta x}$. The symbol used for the slope is m. Here it is 0.5. We can choose a different triangle of Δx and Δy along the line, but their ratio, the slope, remains the same. In fact, we could define a straight line as one whose slope is constant. The equation of this line is $y = 0.5x + 1.5$. In general, the equation of a straight line is $y = mx + b$.

The main part of an equation is the *equal* sign. It tells you that what is on the left of it, the left-hand side of the equation (lhs), has the same value and sign as what is on the right-hand side (rhs). $3 + 8 = 4 + 7$, for instance. Most often there are not just numbers, but also symbols, as in $3x = 12$. Now x has to be equal to 4, or the lhs is not equal to the rhs. We can see what x must be by dividing each side by 3. This leaves x by itself on the left, and $\frac{12}{3}$ or 4 on the right.

We have changed both sides in the same way by dividing each side by the same number, 3. We can also multiply each side by the same number or quantity, add the same thing to each, or subtract the same from each side. In each case the lhs will still be equal to the rhs. The equal sign will still *hold*.

Equations can be much less complicated than expressing what they say in words. In fact, symbols and equations give us a marvelous shorthand way to express complex relationships and to describe a great deal of information.

Each equation describes a relationship: the lhs is equal to the rhs. We can change it in a lot of ways as long as each change preserves that basic relationship. Often we want to make these changes, manipulate the equation, so that one quantity (the *unknown*) is on the lhs, with other quantities, presumably known, on the rhs. If we can do this, we say that we have *solved* the equation.

EXAMPLE 11

The sales at a lemonade stand can be described by the function $y = -(x-2)^2 + 9$, where x is the number of hours of operation after opening and y is the rate of sales in cups per hour.
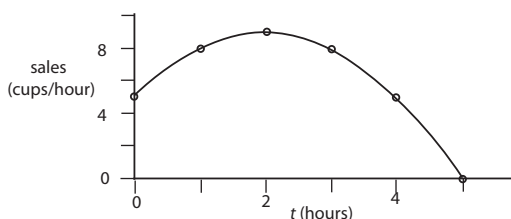
(a) Make a table of the values of x and y at hourly intervals.

(b) Plot the points from your table and sketch $y(x)$.

(c) After how many hours does the number of cups per hour go to zero?

Ans.:

(a)

| x (h)	y $\frac{\text{cups}}{\text{h}}$
0	5
1	8
2	9
3	8
4	5
5	0

(b)

(c) The rate is seen to go to zero after 5 hours.

From the equation: when $y = 0$ it reduces to $0 = -(x-2)^2 + 9$, or $(x-2)^2 = 9$, and, if we take the square root of each side, $x - 2 = \pm 3$, so that $x = 5$ or -1.

Only positive values of t are appropriate here. But you can see on the graph that if the line is continued to the left, it crosses the x-axis at $x = -1$. This shows that it is not enough to follow the mathematics blindly. The solution has to be looked at to see whether it represents the physical situation correctly.

Right-angled triangles

Positive and negative numbers are sufficient to describe positions, distances, and directions as long as we stay along one line, i.e., in *one dimension*. On a plane (in *two dimensions*) or in three-dimensional space we need further tools.
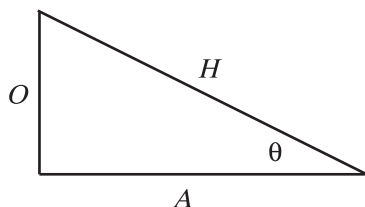
For example, we often make use of the properties of right-angled triangles.

The figure shows a triangle with one right angle (90°) and another angle, θ. One of its sides is marked "O" (for *opposite*) because it is opposite the angle θ and one side is marked "A" (for *adjacent*) because it is adjacent to this angle. The third side is the *hypotenuse*, "H," whose length is given by *the theorem of Pythagoras* (or the *Pythagorean theorem*) $A^2 + O^2 = H^2$.
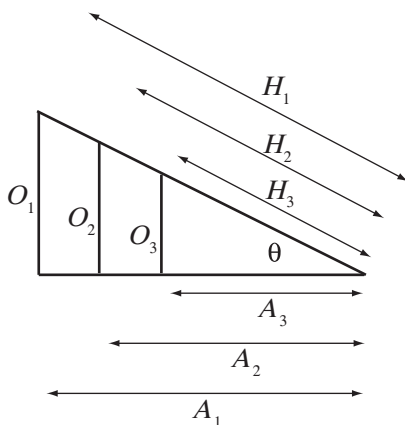
The ratio $\frac{O}{H}$ is the same as long as θ does not change. ($\frac{O_1}{H_1} = \frac{O_2}{H_2}$, etc.) This ratio is called the *sine* of the angle θ: $\sin θ = \frac{O}{H}$. The ratio $\frac{A}{H}$ is the *cosine* of θ (cos θ). The ratio $\frac{O}{A}$ is the tangent: $\tan θ = \frac{O}{A}$.
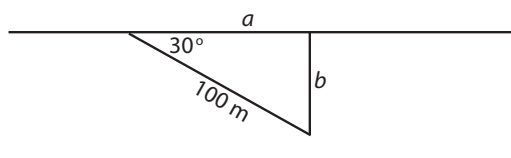
If you divide each term in the relation $A^2 + O^2 = H^2$ by H^2, you get $\frac{A^2}{H^2} + \frac{O^2}{H^2} = 1$, or $sin^2 θ + cos^2 θ = 1$, for any angle θ.

EXAMPLE 12

You swim from shore for 100 m at an angle of 30° with respect to the coastline.

(a) How far are you from the shore when you stop?

(b) If you swim directly back to the shore, how far will you then be from where you started?

Ans.:

The triangle shows the initial part of your path of 100 m, and the second part, to the shore, which is marked b, as well as the distance from there back to the starting point along the shore, which is marked a.

(a) $\sin θ = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{b}{100}$. $b = 100 \sin θ = 50$ m.

(b) The distance back is a.
  $\cos θ = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{b}{100}$. $b = 100 \cos θ = 87$ m.

2.2 Once more the four forces, this time quantitatively

The gravitational force

The relation that describes the gravitational force was first published by Isaac Newton in 1687 in his book *Philosophiae Naturalis Principia Mathematica*, "The Mathematical Principles of Natural Philosophy," usually referred to as *Principia*. It is known as Newton's law of universal gravitation:

$$F_g = G \frac{M_1 M_2}{r^2} \tag{2.1}$$

It describes the gravitational force of attraction between any two bodies with masses M_1 and M_2, separated by a distance r. G is a proportionality constant. In the SI system, where M is in kg, r in m, and F_g in N, G is equal to 6.67×10^{-11} Nm^2/kg^2.

The law is for small "point" objects, but we will see later that it holds also for large spherically symmetric bodies if for r we use the distance between their centers.

EXAMPLE 13

Calculate the gravitational force between the earth and a 1 kg object at its surface.

Ans.:

The first mass, M_1, becomes the mass of the earth, which we can write as M_e, equal to 5.97×10^{24} kg.

M_2 is the 1 kg mass. For r we have to use the distance from the center of the earth to the mass at the earth's surface. This is the radius of the earth, R_e, equal to 6.38×10^6 m. Then $F_g = G \frac{(M_e)(1)}{R_e^2}$, or $\frac{(6.67 \times 10^{-11})(5.97 \times 10^{24})}{(6.38 \times 10^6)^2}$, which comes out to be 9.78 N. Instead of calculating the answer directly it is helpful to collect the exponents separately. Here this is $\frac{(6.67)(5.97)}{(6.38)^2} \times 10^{-11+24-12}$, which is 0.978×10^1 or 9.78 N as before. This makes it easy to check the exponents and to see whether the answer is in accord with a rough estimate. Because the earth is not a homogeneous sphere, nor exactly spherically symmetric, the magnitude of g varies somewhat over the surface of the earth. We usually use the value 9.8 N/kg.

The result shows that the gravitational force that the earth exerts on an object whose mass is 1 kg is 9.8 N. Another way of saying that is that an object whose mass is 1 kg has a weight of 9.8 N at the surface of the earth.

EXAMPLE 14

At what distance from the surface of the earth does an object weigh half as much as it does on earth?

Ans.:
The gravitational force on an object whose mass is M is $F = G \frac{MM_e}{r^2}$. We see that this force (the weight) is proportional to $\frac{1}{r^2}$. At the surface of the earth $r = R_e$. We will call that distance R_1, and the force there F_1.

At some distance that we will call R_2, the force is F_2, so that $\frac{F_2}{F_1} = \frac{1}{2}$. The proportionality shows that $\frac{F_1}{F_2} = \frac{R_2^2}{R_1^2}$, which we can solve for R_2 to give $\sqrt{2}R_1$, or 1.41 R_e. This is the distance from the center of the earth. If we want the distance from the earth's surface, we have to subtract R_e, to get $0.41R_e$ as the answer.
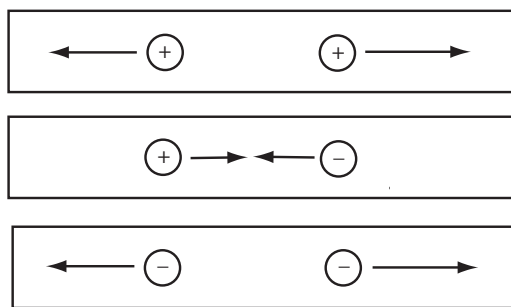

The electric force

Charles Augustin Coulomb did the experiments that led to the law describing the force between charges in the years 1785 to 1787. It is called Coulomb's law, and the force is often referred to as the Coulomb force,

$$F_e = k \frac{Q_1 Q_2}{r^2} \qquad (2.2)$$

Here F_e is the electric force between two point charges, Q_1 and Q_2, a distance r apart.

The SI unit for electric charge is the coulomb. k is a proportionality constant, which in the SI system is 9.00×10^9 Nm²/C². When both charges are positive or both negative, they repel; when one is positive and one is negative, they attract.

EXAMPLE 15

Here is an example that illustrates how enormously strong the electric force is. There are about 6.0×10^{23} atoms in 1 g of hydrogen. Imagine that all of the electrons in 1 g of hydrogen are put on the earth's north pole, and all the protons on the south pole 12.8×10^6 m away. Calculate the force between them.

Ans.:
The charge on a single electron or proton has a magnitude of 1.6×10^{-19} C. In the 1 g of hydrogen the electric charge of the protons is therefore (6×10^{23}) (1.6×10^{-19}) or 9.6×10^4 C. The negative charge on the electrons from the same gram of hydrogen has the same magnitude. The force between them is therefore $F_e = \frac{(9.0 \times 10^9)(9.6 \times 10^4)^2}{(12.8 \times 10^6)^2} = 5.1 \times 10^5$ N, which is equal to the weight of a mass of 5.2×10^4 kg or 52 metric tons.

(Since the weight of a small apple is about 1 N, the force is equal to the weight of about half a million apples.)

The other forces

The gravitational and the electric forces are long-range forces, and each decreases as $\frac{1}{r^2}$, i.e., $F \propto \frac{1}{r^2}$. If the separation of two masses or charges is increased by a factor of two, the force between them decreases by a factor of four. The gravitational force gets smaller with distance, but it still acts over very large distances, such as those between the sun and the earth and the other

planets. The electric force has the same distance dependence.

The nuclear force can not be described by a similarly simple formula. But we know that it acts only between neighboring nucleons in a nucleus, or in collisions when they get just as close to each other.

For the weak force there is also no simple formula. Its contribution is to allow some nuclear processes to take place that could not otherwise happen. It is not responsible for any structures, as is the gravitational force for the planetary system, the electric force for atoms, and the nuclear force for nuclei.

2.3   Summary

Mathematics is the language of physics. Here are some reminders and reviews of some parts that we are going to use.

A *symbol* is shorthand for a quantity (like x) or an operation (like $+$). Symbols allow us to express relationships between quantities by *equations*.

Numbers can be positive, negative, or zero. Very large and very small numbers are best expressed with powers of ten. 5×10^6 is the same as 5 followed by six zeros, or 5 million. The *exponent* (in this case 6) gives the number of spaces by which the decimal point is shifted—to the right for positive exponents, and to the left for negative ones.

The number of *significant figures* indicates how precisely a number is known. When we give no other information about it we will assume that the numbers in the examples and problems are known to three significant figures.

Physical quantities are expressed in terms of *units*. We will most often use the units of the SI system, which include meters, (m), kilograms (kg), and seconds (s).

Relations between quantities can be described by using *tables, graphs*, or *equations*. All of these are *representations* of what happens, as are words and pictures.

A straight line on a graph of y against x is described by the equation $y = mx + b$. y is usually plotted along the vertical axis and x along the horizontal axis. m is the slope, equal to $\frac{\Delta y}{\Delta x}$, and b is the y-intercept, i.e., the value of y when x is zero.

In a triangle with a right angle and an angle θ, with O the side opposite θ, A the side adjacent to θ, and H the hypotenuse (whose length is $\sqrt{O^2 + A^2}$), $\sin \theta = \frac{O}{H}$, $\cos \theta = \frac{A}{H}$, and $\tan \theta = \frac{O}{A}$.

Newton's law of gravitation, $F_g = G \frac{M_1 M_2}{r^2}$, describes the gravitational force between two particles with masses M_1 and M_2, a distance r apart. Coulomb's law, $F_e = k \frac{Q_1 Q_2}{r^2}$, describes the electric force between two particles with charges Q_1 and Q_2 a distance r apart. The two expressions are similar in that both forces are inversely proportional to the square of the distance between the two interacting particles. Each includes the product of the characteristic quantities (mass or charge) of the particles. Each expression describes a very different physical property. Each has a very different strength, reflected in the respective proportionality constant (G or k).

2.4   Review activities and problems

Guided review

1. You walk to the right 50 yards, then to the left 17 yards.

(a) Represent the motions mathematically, letting "to the right" be positive. What does the result and its sign tell you?

(b) Now answer the question again, but this time letting "to the left" be positive. What does the result and its sign tell you?

2. The temperature drops from 15°F to 0°F.

(a) Write a mathematical description of what happens.

(b) Convert the temperatures to the Celsius scale and repeat.

3. A sign whose weight is 100 lbs is held up by a chain.

(a) With what force does the chain need to be pulled up to keep the sign in place?

(b) Write a mathematical description.

(c) What does the sign of each number represent?

4. Two properties of several elements are shown in the table. One is the atomic number and the

other is the most common ionization state. The number in the superscript shows how many electrons have been added to the neutral atom or removed from it. A "+" sign indicates that the resulting ion is positive and a "−" sign that it is negative.

(a) How much positive and how much negative charge is in each neutral atom? (Give the answer in multiples of e, the magnitude of the charge on an electron.)

(b) Similarly represent the charges in each of the ionized states.

Element	Al	Zn	N	C
Atomic number	13	30	7	6
Common ionization state	Al^{3+}	Zn^{2+}	N^{3-}	C^{4-}

5. The mass of a proton or a neutron is 0.000 000 000 000 000 000 000 000 001 67 kg. The mass of an electron is 0.000 000 000 000 000 000 000 000 000 000 911 kg. Put these numbers in power-of-ten notation.

(a) If you put a proton on one pan of a balance, how many electrons would you have to put on the other to balance it?

(b) What is the total mass of the particles that make up a neutral helium atom?

6. The circumference of a circle is 2.735 m. How big is the diameter? (How many significant figures do you know?)

7. Convert the following into SI units. (1 in = 2.54 cm, 1 mile = 5280 ft, 1 food calorie = 1000 cal = 4187 J, 9.8 N = 2.20 lb)

3.4 cm

5 in

140 lbs

30 $\frac{miles}{hour}$

14.7 $\frac{lb}{in^2}$

2000 $\frac{food\ cal}{day}$

8. The strength of attraction, F_m, between two magnets can be described as $F_m \propto \frac{1}{d^2}$.

You separate the two magnets from $d = 1$ cm to $d = 5$ cm. By what factor does F_m change? Does it increase or decrease?

9. Here is a table of the amount of water (in gallons) consumed by a soccer team during a practice session on a hot afternoon as a function of time in minutes.

Time	30	60	90	120	150	180
Water consumed	1.25	2.5	3.75	5	6.25	7.5

(a) Make a graph of the data.

(b) Write a mathematical function that describes the data.

(c) What is the rate at which the team drinks?

(d) They start with 10 gallons. How long can they practice before they run out of water?

10. Here is a plot of the number of leaves on a tree as a function of time, in days after the leaves have started to fall.

(a) Write an equation that describes the data.

(b) After how many days will all of the leaves have dropped if the equation continues to be valid?
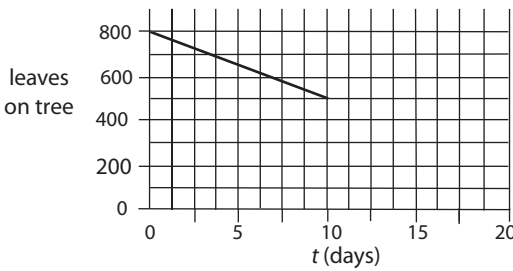
11. On a very hot day the sales at a lemonade stand can be represented by the equation $y = (x - 2)^2 + 9$, where y is the rate of sales in cups per hour and x is the number of hours of operation after opening from $t = 0$ to $t = 5$, at which time all of the lemonade has been sold.

(a) Make a table of values of x and y at hourly intervals.

(b) Plot the points and sketch $y(x)$.

(c) What is the slope at $t = 2$ and 4?

12. A flagpole is mounted on the side of a house. It is 1 m long and makes an angle of 50° with the vertical.
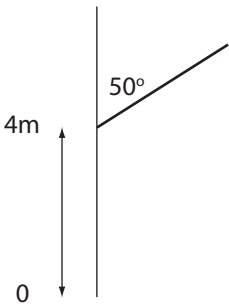
(a) If the sun shines straight down, how long is the flagpole's shadow?

(b) How far from the ground is the end of the flagpole?

13. What is the gravitational attraction between two 60 kg people standing 2 m apart?

14. What is the weight of a 1 kg mass on top of Mount Everest? ($H = 8850$ m)?

(a) Use proportional reasoning and the fact that the weight is 9.8 N at sea level.

(b) Check your answer by using Newton's law of gravitation.

15. Two spheres are electrically charged, one with $Q_1 = -15$ nC and the other with $Q_2 = +15$ nC. (1 nC $= 10^{-9}$ C.) This is the charge of about 100 billion electrons and protons. Find the electric attraction between the two spheres when they are 2 m apart.

Problems and reasoning skill building

1. You are out skiing, and in the course of the afternoon the temperature drops by $8°C$ from a beginning temperature of $4°C$. Describe the temperature change using integers. What does the result and its sign tell you?

2. Two cylindrical buildings are side by side. One has a diameter of 100 m and a circumference of 314 m. The other has a diameter of 135 m. Use ratios only (not π!) to find the circumference of the second building.
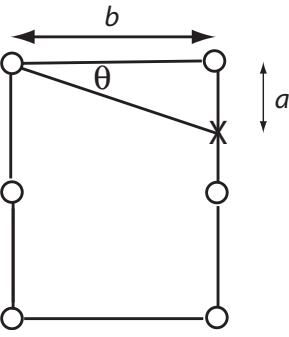
3. The data in the table show how many pushups a young boy completes over a period of five days.

Day	Pushups
1	4
2	7
3	10
4	13
5	16

(a) Make a graph of the data.
(b) Write a function that describes the data.

4. The figure shows a pool table. Your ball is at the position marked X. $a = 12''$, $b = 36''$.

(a) How far is your ball from the left top corner pocket?

(b) What is the angle θ?

5. At the time of his famous studies on electricity Benjamin Franklin understood that there were two kinds of electric charge. He called one kind *positive* and the other *negative*. According to his (incorrect) model electric current in a wire is the movement of positive charge. The current, I, is defined as the amount of positive charge passing a cross section of the wire per second. The figure shows 15 positive charges in a wire, which move to the right, past the place marked by a dotted line in one second.
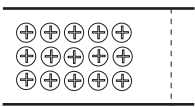
Today we know that it is actually the negative charges that move. What is the motion of charge for the same amount of current to the right, when negative charges move?

6. Your brother raises gerbils. At the end of each month you count the baby gerbils in the cage. The table shows how many there are for the first five months.

Month	Babies
1	0
2	3
3	8
4	15
5	24

(a) Make a graph of the data.
(b) Find a function that describes the data.

(c) If this trend continues, how many babies do you expect to find at the end of the sixth month? The seventh month?

(d) What factors does your mathematical model not consider?

7. Some people believe that the position of the planets at one's birth has a profound influence on the course of a person's life.

Calculate the gravitational interaction with a newborn of the planet Jupiter and of the attending physician. Which is larger?

8. The angle that the line to the sun makes with the horizontal can be used to estimate the time to sunset. (In the afternoon the angle can be estimated from the fact that $\sin 30° = 0.5$.) What is the time to sunset when the sun is $15°$ above the horizon?

9. A measure of the apparent size of the moon is the angle that it subtends, i.e., the small angle in the triangle of which the moon's diameter is one side and the two other sides are equal to the distance to the moon. The angle can be estimated by holding a pencil or piece of paper at arm's length so that part of it forms a triangle similar to the first, with the same small angle.

Draw a diagram with both triangles. Estimate the angle, and use the distance to the moon to estimate the diameter of the moon. Compare it to the value in a table of astronomical quantities.

10. Convert the following energy units to joules. Give the answer to three significant figures and use power-of-ten notation with one digit before the decimal point. Put the answers in the order of increasing energy.

cal
kwh
horsepower hours
eV
MeV
ft-lbs

11. The weight of a rock on earth is 50 N.
    (a) What is its mass?
    (b) Use Newton's law of gravitation to calculate the value of g on the moon.
    (c) What are the mass and weight of the rock on the moon?

12. What can you conclude from the fact that the apparent sizes of the moon and the sun are approximately the same?

13. Use Newton's law of gravitation to determine the SI units of G. Use Coulomb's law to determine the SI units of k.

14. The tides are caused by both the moon and the sun. Which of the two exerts the greater force on the earth, and by what factor?

15. What is the ratio of the gravitational force between the electron and the deuteron in a deuterium atom to that in ordinary hydrogen (whose nucleus is a proton)? Answer the same question for the electric force and for the total force.

16. Use proportional reasoning to find by what factor the energy of motion of a runner, $\frac{1}{2}mv^2$ (where v is her speed), increases when she improves her racing time for a 5 km course from 22 min to 19 min.

17. A medieval alchemist had high hopes of turning lead into gold. The densities of the two metals are $\rho_{gold} = 19.3 \times 10^3 \text{ kg/m}^3$ and $\rho_{lead} = 11.3 \times 10^3 \text{ kg/m}^3$.

If he had succeeded, what would be the ratio of the space occupied by 1 kg gold to that occupied by 1 kg of lead? What would be the difference in the amounts of space?

18. An accused thief, named Stiles, implores Sherlock Holmes to help him, saying that he is wrongfully accused of stealing one million dollars in gold. The evidence against him is that he was seen placing the gold in a suitcase and then escaping, running down the street with the missing gold. Holmes asks Watson "How much does an ounce of gold go for these days?" and is told that it is 200 British pounds, with a value of 300 U.S. dollars. Why does Holmes agree to take the case?

Synthesis problems and projects

1. At "half moon" we see the moon illuminated by the sun from the side.

(a) Draw a diagram that shows the positions of the earth, the moon, and the sun at that time. What is the angle beteen the line EM from the earth to the moon and the line SM from the sun to the moon?

(b) Aristarchus of Samos (310 B.C.–230 B.C.) measured the angle θ between the line EM and the line ES from the earth to the sun. From this measurement he calculated the ratio of the length of

ES to the length of EM. How is this ratio related to the angle θ?

(c) From your knowledge of the modern values of these lengths (see the table at the end of the book) what are the values of the ratio and of the angle?

(d) Look up the value of the angle measured by Aristarchus and the ratio that he calculated.

(e) Why does a small error in the estimate of the angle lead to a large error in the ratio of the two distances?

(f) Aristarchus also noted that the angles subtended by the moon and the sun at the earth (i.e. their apparent sizes) are approximately the same. What does this observation lead to for the ratio of the radii of the two bodies? How does his value compare to the modern ratio?

2. Aristarchus used observations during an eclipse of the moon to determine the ratio of the diameters of the earth and the moon.

(a) Draw a diagram of the positions of the sun, the earth, and the moon during an eclipse of the moon. (The moon is in the earth's shadow so that it is not illuminated by the sun.)

(b) Aristarchus measured the time interval from the time when the moon first comes to the earth's shadow to the time when it is completely in the earth's shadow. He also measured the time of "totality," i.e., the time during which the moon is completely in the earth's shadow. This second time interval turned out to be approximately twice the first. Indicate the approximate size relation of the moon and the earth on your diagram.

The geometrical calculation of the ratio of the diameters from the observation is somewhat complicated. Several versions may be found on webpages devoted to Aristarchus. There you will also find discussions describing that he seems to have been the first to believe that the earth revolves around the sun rather than the other way around. Copernicus, more than 1700 years later, is usually given the main credit, but he refers to the work of Aristarchus.

3. Eratosthenes of Cyrene (273 B.C.–192 B.C.) determined the circumference of the earth as follows. At Aswan, on the Tropic of Cancer, at the summer solstice, at noon, there is no shadow in a vertical well, i.e., the sun is directly overhead. At that time, at a distance of 4900 stadia further north, in Alexandria, (with one stadium probably about 160 m) the sun was at an angle of 7° away from the vertical.

(a) Draw a diagram to show the earth and the sun at that time in the two places.

(b) What does his measurement lead to for the radius of the earth? What is the modern value?

4. The half lives of ^{238}U and ^{235}U are 4.5×10^9 y and 0.71×10^9 y, respectively. Consider the hypothesis that the two isotopes of uranium were present in equal amounts at a time near the beginning of the universe. Today ^{235}U is 0.72% of natural uranium. The rest is ^{238}U. Determine the time implied by this hypothesis as follows.

Make a graph of the amount of ^{235}U as a function of time on which the horizontal axis shows time in units of 10^9 y. On the vertical axis let equal factors be represented equally, i.e., one unit from 2 to 4, the next from 4 to 8, then from 8 to 16, and so on. (This is called a semilog plot.) On this graph the amount of uranium as a function of time is a straight line.

5. An electrically charged ball with a mass of 20 g floats 10 cm above a second one that carries the same charge, q, so that the net force on the floating ball is zero. What is the value of q?