CHAPTER 3

# There Is No Rest: Describing Motion

Getting started: simplification and approximation: models

Keep your eye on the ball: where is it and where is it going?

*How far? Distance and displacement*
*How fast? Speed, velocity, and acceleration*
*Constant velocity*
*Changing velocity: constant acceleration*

The mathematics of change

*Slopes and derivatives*
*Areas and integrals*

There was a time when physics meant *mechanics*, and perhaps it should still. The word comes from the Greek for machine. We understand a piece of machinery when we know each of its parts, all the gears and levers, and their functions and interplay. That's really the program of all of natural science, to know the pieces of which the world is made and how they connect and work together.

The world is more complicated than any machine, and knowing it and understanding it is a work in progress. We learn more each day, we get to know some parts very well, but there is no end to the quest. We continue to be surprised by what we learn, sometimes by the complexity and intricacy, sometimes by the simplicity of the way in which the components act and interact.

The term *mechanics* has come to be used in a more restricted sense, as the science of motion and of the way that forces bring about motion. To start with, we will limit ourselves further by leaving out, at one end of the scale, the atoms and their constituents, and at the other end the stars and other massive astronomical objects, because they behave, in part, in ways that are different and outside our day-to-day experience. That leaves us with *classical* mechanics, the study of the realm between, the ordinary world of chairs and tables, baseballs and bullets.

We will look at the basic concepts that we need to describe motion, namely displacement, speed and velocity, and acceleration.

One of the features that has made physics so successful is that it asks the simplest questions, and so is able to look at them in great depth. That is also one of the dangers. Physicists love to go into more and more detail, until what they say is far from simple. That can easily happen when we go into the mathematical description of various kinds of motion, along straight lines, in circles, parabolas, ellipses, motion in one dimension or in many, rotational, spiraling, oscillating, and so on.

We will do only a little of that. The main reason we study mechanics is to see how it introduces basic ideas, such as force and energy, that are central to every part of science. These ideas will stay with us as we go on later to other realms, beyond the world that is immediately accessible to us through our senses. There we come to atoms and molecules, each one of them ceaselessly moving. It was one of the great insights in intellectual history when it was realized that it is their motion that underlies, on the larger scale, all of our experience with the phenomena of heat and temperature. It is also here, in the world of atoms and molecules, that we will find the forces between them, and within them, that shape the animate and inanimate world as we know it in its infinite variety.

## 3.1 Getting started: simplification and approximation: models

Look at yourself running. Your elbows move. Your feet move differently. In fact, every part of you moves, and moves differently from every other part. Usually you just see yourself as running in a straight line at a certain speed, and don't think about these much more complicated motions.

When you think only of the fact that you are running, concentrating on the whole of you, ignoring the fact that your knees are moving quite differently from your chin, the picture in your mind becomes much simpler. You can then describe your motion quite straightforwardly by saying "I am moving along this track with a speed of 6 miles per hour."

Now think of all that you have decided to ignore: not only the externally obvious parts of your body, all moving differently, but also the internal, vastly complex motions, as you breathe, swallow, digest, as your heart beats and your blood flows. On an even deeper level, each part of you consists of molecules, and each molecule is moving. Inside the molecules, the atoms are also moving, and inside the atoms the electrons and nuclei.

With each step that you take, you are pushing on the ground, changing its motion, and at least in some minute way you are changing the motion of the earth of which it forms a part, as it spins and travels around the sun, the earth and the sun each a collection of vastly complex constituents, all of them ceaselessly moving.

It's a good thing that you can simplify the story, ignoring most of it, and concentrating just on your motion around the track. That's what allows us to describe what is going on without being overwhelmed, without being paralyzed by the difficulty of dealing with every detail at once. This simplification is an essential part of science, and makes it possible for knowledge to progress, piece by piece.

You can study the motion of an object simpler than a human being, perhaps a baseball as it flies through the air. But here too you will have a tough time unless you begin by ignoring a large part of what happens. You may want to restrict yourself to still air, but even then, as the air molecules push against the ball, they will affect its motion. Internally, the ball is made of different materials, and they, like everything else, are made of moving molecules, atoms, and nuclei.

Can you imagine something even simpler, with no internal structure? Yes, at the smallest level there are "elementary particles," so called because we think that they have no internal structure at all. On our own much larger scale we can talk about balls and marbles as if they were particles. To do so we have to ignore a lot, and keep in mind that we are no longer talking about the ball or the marble as it exists. We are substituting something that we have invented, something that we will call a *model* of the ball. (This is a very different use of the word "model" than those that you are likely to be familiar with.)

We can talk about the model, in this case a particle, and how it moves and behaves. It is much simpler than the ball. It does not have all the properties of the ball. A particle can't be squeezed, and it can't be heated, because both of these processes represent internal changes. By using the model of the ball as a particle we have given up any possibility of describing what happens inside the ball.

We have to be careful when we simplify and approximate, i.e., when we use a model, because some part of the real story is then lost. But if we are clever, or lucky, the essential part will still be there.

## 3.2 Keep your eye on the ball: where is it and where is it going?

Here is what has been called *the* problem of mechanics: consider a particle at some instant of time. You know where it is, and how fast and in what direction it is moving. You also know the forces that are acting on it, at the beginning, and at all other times. Calculate where it will be and how fast and in what direction it will be moving at some given future moment.

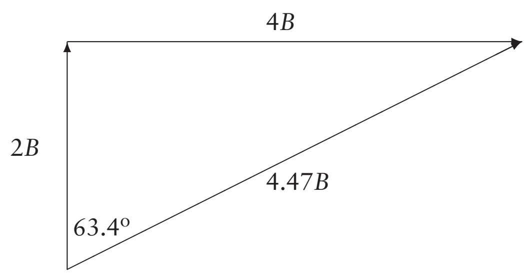
Perhaps the most interesting point is that this problem *has* a solution. Always, as long as the relations of *classical mechanics* hold. What are these relations? Are they *right*? Or better, to what extent, and under what circumstances are they right? Should we think of them as *laws* that have to be obeyed? As we examine these questions and their implications we see already that it is just as important to look at the limits of validity of these or any other relations and "laws" as it is to look at the relations themselves.

We'll start with the description of the motion. This part of mechanics is called *kinematics*. It has to do with position and time, velocity, and acceleration. We will assume that we know what is meant by position and time. It took the genius of Einstein to show that even these concepts cannot be taken for granted. He looked more deeply at mechanics, and came to revolutionary conclusions, which we will discuss later. For now, we'll start at the beginning.

### *How far? Distance and displacement*

When Jane walks to school, she changes her position. She first goes two blocks north along her street, then turns the corner and goes four blocks east. The distance she goes is therefore six blocks. If each block is 500 feet long, that distance is 3000 feet. To go there and back she has to traverse a distance of 6000 feet.

We will define another measure of how far she is from her starting point, called the *displacement*. Its magnitude (or size) is that of the straight-line distance from the starting position to the final position. In this case it is $\sqrt{2^2+4^2}$ blocks, or 4.47 blocks, which is equal to $2.22 \times 10^4$ ft. Besides this magnitude, the displacement has another attribute, namely its *direction*. In the present case it can be specified as 63.4° east of north.

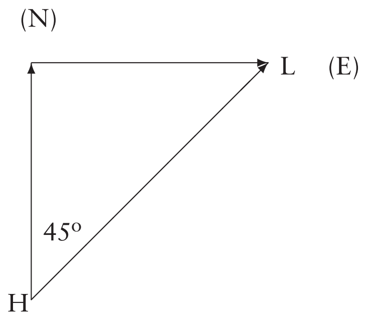



A quantity that has both magnitude and direction is called a *vector* quantity. A quantity that has only magnitude (as, for example, temperature, or volume) is called a *scalar* quantity.

When Jane goes to school and then back, she has gone a distance of 6000 feet. But since the displacement is the distance between the starting position and the final position, it is, in this case, zero.

EXAMPLE 1

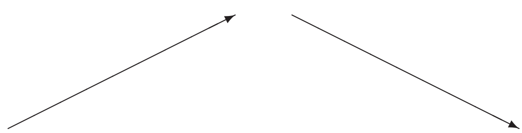
Bob walks to the library. He walks one block north and then one block east. Find the magnitude and direction of his displacement.




Bob starts from home (H) and goes to the library (L). His displacement is the vector from H to L. Its magnitude is $\sqrt{1^2 + 1^2}$, equal to $\sqrt{2}$, or 1.41 blocks. Its direction is 45° east of north.

## *How fast? Speed, velocity, and acceleration*

Speed is what the speedometer shows. If it reads "20 miles per hour" the car will travel a distance of 20 miles in an hour, as long as the speed doesn't change. Usually, though, the speed does change. But if the car traverses 20 miles in one hour, even if it has stopped at a red light, and then speeded up to 35 miles per hour for part of the trip, the *average speed* is 20 miles per hour.



These are two different vectors. They have the same magnitude, but different directions.

If the car goes around a corner, and the speedometer still shows 20 miles per hour, its speed doesn't change. But to a passenger it feels quite different from what it would if the car continued in a straight line. To distinguish between these two situations, we define a different quantity, the *velocity*. Its magnitude is equal to the speed, but the velocity is constant only if the direction does not change. If, for example, a car goes first in one direction and then back in the opposite direction, both at 20 miles per hour, the speed is the same on the return trip, but the velocity is not. The speed has only magnitude. It is a scalar quantity. The velocity has magnitude and direction. It is a vector quantity.

As the car moves, its displacement from the starting point changes. On a graph the displacement, $x$, is measured from an *origin*, where $x = 0$. If $x$ is plotted against $t$, we call this a graph of $x(t)$ (read "$x$ as a function of $t$" or "$x$ of $t$").

The quantity that describes how the displacement changes is the velocity. It is the rate of change with time of the displacement. When the velocity changes, there is an *acceleration*. The acceleration is the rate of change with time of the velocity. We will explore the relations between displacement, velocity, and acceleration in the rest of this chapter.

EXAMPLE 2

Bob walks one block north in five minutes.

(a) At what rate and in what direction does his displacement change?

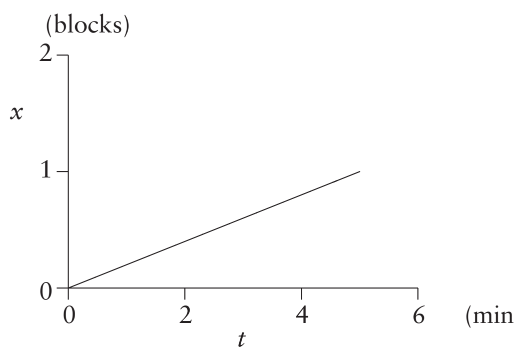
(b) Draw a graph of $x(t)$ for Bob's motion.

(c) Find the slope of the graph and compare it to the answer to (a).

*Ans.:*

(a) The rate at which the displacement changes is the velocity. It is $v = \frac{1\text{ block}}{5\text{ min}} = 0.2\ \frac{\text{blocks}}{\text{min}}$ to the north.

(b)



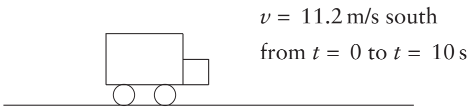

(c) The graph shows that the slope is $\frac{1\text{ block}}{5\text{ min}} = 0.2\ \frac{\text{blocks}}{\text{min}}$. The slope of $x(t)$ is the velocity.

## *Constant velocity*

Let's talk about a car that moves in a straight line with a velocity of 25 miles/hour south for 10 seconds. This is the *verbal representation* of the situation that we want to describe.

First of all, we prefer the SI units of m/s: 1 mile is about 1609 m, so that $\frac{25\,\text{miles}}{\text{h}} = \left(\frac{25\,\text{miles}}{\text{h}}\right)\left(\frac{1609\,\text{m}}{1\,\text{mile}}\right)\left(\frac{1\,\text{h}}{3600\,\text{s}}\right) = 11.2\,\frac{\text{m}}{\text{s}}$.

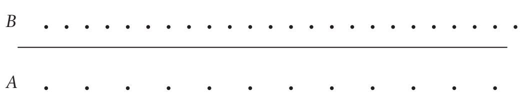
In addition to the verbal statement we can represent this motion in several other ways. We can draw a picture. This will be a snapshot at a particular moment. By itself it won't tell us anything about the motion, but we can write on it whatever we know about the motion. In this case we know that the velocity is $v = 11.2$ m/s south, during the time interval from $t = 0$ to $t = 10$ s.




We can go further by drawing several snapshots at equal time intervals, as on a movie film. In this case, with its constant velocity, the car will move 11.2 m in each second, i.e., the same distance in each of the equal time intervals. We will call the sequence of such snapshots a *motion diagram*. To keep things simple we will represent the car by a dot. This is in keeping with the model of the car as a particle. Usually our motion diagrams will be qualitative, i.e., they will not have numbers or units, and just show whether the dots are, as in this case, equally spaced, or whether they get closer together or farther apart.



EXAMPLE 3




(a) Which car moves faster, *A* or *B*?

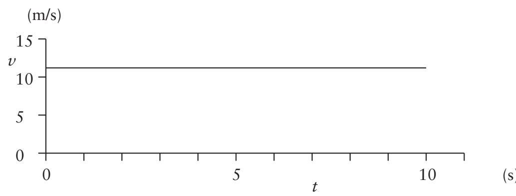
(b) Are they ever side by side? If so, where?

*Ans.:*

(a) The points in *A* are farther apart. That means that the car moves through a greater distance in the same time. This car goes faster.

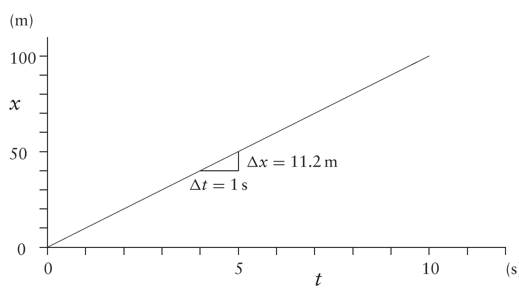
(b) They start out at the same place and are side by side at the beginning. After that, *B* goes more slowly, and the cars are never side by side again.

We can represent more information on a graph, or on a series of graphs. Look first on a graph of $v$ against $t$. The velocity is constant. It has the same value for each value of $t$. Another way of saying the same thing is that the velocity as a function of time, $v(t)$, is constant. As $t$ varies, $v$ remains the same. The graph is therefore a horizontal line.




Note that the graph is marked with the quantities being plotted, $v$ on the vertical axis and $t$ on the horizontal axis. On each axis we also mark the units, m/s on the vertical axis and s on the horizontal axis.

What about the displacement, $x$? It starts at $x = 0$ when the time is $t = 0$. It then increases by 11.2 m every second. The graph is a straight line, but it is not horizontal. Again we mark the quantities and units, $x$ and m on the vertical axis and $t$ and s on the horizontal.




To see how fast $x$ varies with $t$, we look at a change in $t$, for which we use the symbol $\Delta t$ (read "*delta t*"). Let's make this time interval 1 s. During this time there is a change in $x$, which we call $\Delta x$, of 11.2 m. The slope of the graph is $\frac{\Delta x}{\Delta t}$, equal to 11.2 m/1 s, or 11.2 m/s. This is the velocity. The velocity has the units of $x$ divided by the units of $t$, or m/s.

We can write a mathematical representation that describes this line. (We could also call it a "mathematical statement," or an "equation.") We can do that by starting with the standard expression for a straight line, $y = mx + b$. In this expression $y$ is plotted along the vertical axis, $x$ along the horizontal axis, $m$ is the slope, and $b$ is the $y$-intercept. (That's the value of $y$ when $x$ is equal to zero.) We can compare that with our present graph, where we plot the displacement along the vertical axis and the time along the horizontal axis. The slope is the velocity and the intercept is zero. The expression becomes $x = vt + 0$, or $x = vt$.

EXAMPLE 4

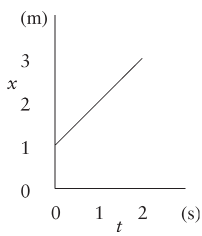
The graph describes Jean walking west.

(a) Where is she at $t = 2$ s?

(b) What does the intercept on the vertical axis represent?

(c) What is the slope of the line? What physical quantity does it represent?

(d) Write a mathematical representation of the line, first with symbols and then with numbers.




*Ans.:*

(a) The distance, $x$, is measured from a chosen origin, with "west" the positive direction. At $t = 2$ s she is 3 m west of the origin.

(b) The intercept is her position at $t = 0$, which is $x_0$. At that time she is 1 m west of the origin.

(c) The slope is her velocity. It is 1 m/s to the west.

(d) Comparing to the standard expression for a straight line,

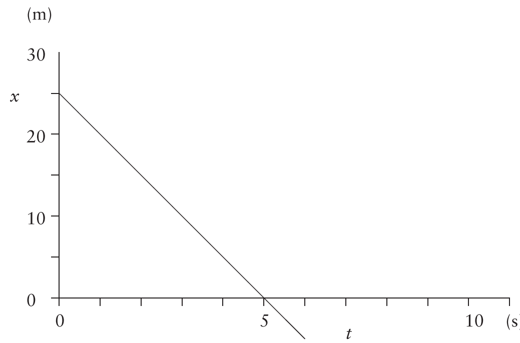
$$y = mx + b$$

we have to change $y$ to $x$, $(y \to x), x \to t, m \to v, b \to x_0$, so that the mathematical expression becomes

$$x = vt + x_0$$

With the numbers, her displacement in meters, in terms of the time in seconds, is $x(\text{m}) = \text{t}(\text{m/s} \times \text{s}) + 1(\text{m})$, or $x = t + 1$, where the units of each term are meters (m).

EXAMPLE 5




The graph shows the motion of a car as $x(t)$ for 6 s. The $x$ direction is toward the right. The negative $x$ or $(-x)$ direction is to the left.

(a) Where is the car at $t = 0$?

(b) Where is the car at $t = 5$ s?

(c) Where is the car at $t = 6$ s?

(d) In what direction does the car move?

(e) How far does it move between $t = 5$ s and $t = 6$ s?

(f) What is the car's velocity?

(g) What happens at $t = 5$ s?

*Ans.:*

(a) At $t = 0$ the car is 25 m from the origin in the $+x$ direction. The car is 25 m to the right of the origin.

(b) At $t = 5$ s the car is at the origin, $x = 0$.

(c) At $t = 6$ s the car is 5 m from the origin, in the negative $(-x)$ direction, or 5 m to the left of the origin.

(d) The car moves in the $-x$ direction, toward the left.

(e) It moves 5 m in the negative $x$-direction, to the left.

(f) It is moving with a constant speed of 5 m/s in the negative $x$ direction, i.e., to the left. Its velocity is $-5$ m/s.

(g) At $t = 5$ s the car passes the point $x = 0$.

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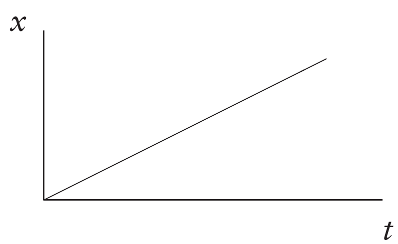
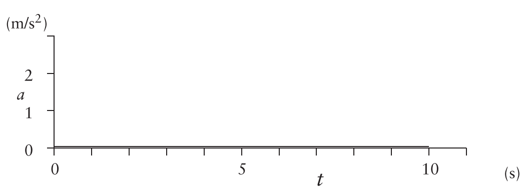
These graphs are quantitative. They show the magnitudes of $v$ and $x$ for each value of $t$. Sometimes we may want to draw a qualitative graph or *sketch* that just shows the shape of each function. Here, for example, is a sketch of $x$ against $t$. The axes are still marked with $x$ and $t$ to show what quantities are being plotted, but without the units and numbers.




We can also draw a graph that shows how the acceleration, $a$, changes as the time, $t$, changes. Since the velocity does not change, the acceleration is zero. A graph of $a$ against $t$ is therefore a horizontal line at the value $a = 0$, right at the horizontal axis.




The units of $t$ are seconds, but what are the units of the acceleration? If the velocity changes from 5 m/s to 11 m/s in 2 s, the change in the velocity ($\Delta v$) is 6 m/s. In each second the velocity changes by 3 m/s. The acceleration is 3 m/s per second, or $3\frac{\text{m}}{\text{s}}$/s, which we write as 3 m/s$^2$.

In symbols, the acceleration is the change in the velocity, $\Delta v$, divided by the time interval $\Delta t$ during which the velocity changes by $\Delta v$: $a = \frac{\Delta v}{\Delta t}$. It therefore has the units of $v$ divided by the units of $t$, or m/s divided by s, which is equal to m/s$^2$.

Finally, we can represent the motion by one or more mathematical relations. In this case they are $a = 0$, $v = 11.2$ m/s, and $x = vt = (11.2t)$ m.

To summarize: motion with constant velocity can be described in several ways:

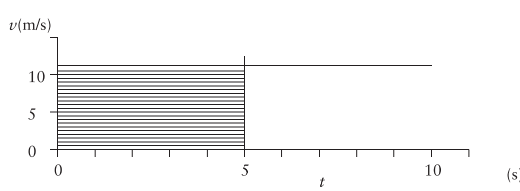
(1) The verbal statement that the object moves through equal distances in equal time intervals.
(2) The motion diagram, which shows the equal distance intervals.
(3) The graphs: $a(t)$, horizontal at $a = 0$; $v(t)$, horizontal at the value (11.2 m/s) of the velocity; and $x(t)$, the straight line through the origin, with slope $v$.
(4) The mathematical expressions: $a = 0$, $v = +11.2$ m/s, and $x = vt$, or $x = (11.2)(t)$ m.

In this case, where the velocity is constant, the slope of the graph of $x$ against $t$ is constant, and the graph is a straight line. But the definition of the velocity as the slope on a graph of $x$ against $t$ still holds when the velocity is not constant.

As you look at the three graphs of $a, v$, and $x$ against $t$, you see that each shows the slope of the next one. $a$, here equal to zero, is the slope of the horizontal line of $v(t)$. The value at which this horizontal line is drawn (11.2 m/s), in turn, is the slope of $x(t)$.

You can also see that the *area* under the graph of $v(t)$ gives the displacement, $x$. The area under the curve for the first second is (11.2 m/s)(1 s), or 11.2 m. This is the value of $x(t)$ when $t$ is 1 s. Note that lines on this graph have units, m/s if they are vertical and s if they are horizontal. A rectangular area therefore has the units of m/s multiplied by s, (m/s)(s), or meters.

For the first five seconds the area is (11.2 m/s)(5 s), equal to 56 m.




The relationships between the slopes and areas will always be the same, regardless of the actual values of $a, v$, and $x$, because they are the direct consequences of the definitions of these three quantities.

EXAMPLE 6

Go to the PhET website (http://phet.colorado.edu) and open the simulation "The Moving Man."

Play with the *Introduction*. Than go to *Charts*.

The position can be changed with the slider or by putting the desired position into the box and then pushing the button at the bottom. The scales can be changed with the buttons marked "+" and "−."

(a) Clear, put $x = -10$ m, $v = 2$ m/s (leave $a = 0$) and push to go. Stop the motion after the man hits the wall. The motion can be repeated by using *Playback*.

What does the graph of $v(t)$ show?

What happens to it when the man gets to the wall?

What is the equation for $v(t)$ before he hits the wall?

(b) Answer the same questions for $x(t)$.

What is the slope of the line?"

(c) What does the graph of $a(t)$ show?

(d) Experiment with other values. Go to "Special Features" and choose the "Expression Evaluator." It shows the graph for a mathematical expression that you put in the box. You have to use "*" for multiplication and "/" for division. Start with $x = 5+2*t$ and experiment with other expressions. Put in your expression for $v$ and compare it to your result in (a).

*Ans.:*

(a) The graph is a straight horizontal line at $v =$ 2 m/s until he gets to the wall, when it goes to zero. The equation is $v = 2$ m/s.

(b) The graph is a straight line. It starts at $x = -10$ m and goes to $x = 10$ m. Its slope is 2 m/s. This is equal to the constant velocity. The equation is $x = -10+2t$. To check, we see that at $t = 0$ $x = -10$ and at $t = 10$ $x = 10$.

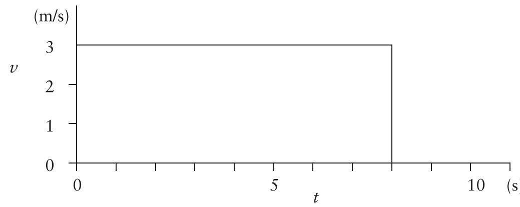
(c) $a(t)$ stays at zero until it shows a negative spike where the man stops abruptly when he hits the wall.

EXAMPLE 7

The graph shows the velocity of a car for 10 s. It first moves at a steady velocity for 8 s.

(a) What is that velocity?

(b) What happens at $t = 8$ s?




(c) How far is the car at $t = 0$ from its position at $t = 8$ s?

(d) How far does the car move between $t = 0$ and $t = 10$ s?

*Ans.:*

(a) The velocity is 3 m/s.

(b) At $t = 8$ s the velocity abruptly goes to zero. The car comes to an immediate stop. It apparently crashes into a wall.

(c) $x = vt = (3\text{ m/s})(8\text{ s}) = 24$ m.

(d) Between $t = 8$ s and $t = 10$ s the car does not move at all. During the 10 s it moves 24 m.

## *Changing velocity. Constant acceleration*

We can use the same representations for other kinds of motion. Look next at what happens when the velocity changes, but the acceleration is constant.

An example is the motion of a freely-falling body near the surface of the earth. By "freely falling" we mean that we consider only the influence of the earth, and assume that the influence of air resistance and other extraneous retarding effects is so small that they can be neglected. That's a good approximation if we drop a metal object like a coin, but not for a piece of paper or a feather. It works very well on the moon, where there is almost no atmosphere.

It may only be legend that Galileo, around 1590, dropped objects with different weights from the leaning tower of Pisa, and showed that they all take the same time to fall to the ground. In any case, he showed in his later writings that free fall was motion with constant acceleration.

The magnitude of the acceleration of a freely-falling body is so important that we use a special symbol for it, $g$. Near the surface of the earth its magnitude is approximately 9.8 m/s$^2$, or 32 ft/s$^2$, and it is in the downward direction. (The speed increases by 32 ft/s in each second.) If you go up a mountain, $g$ decreases. On the moon a freely-falling object falls with an acceleration of only 1.6 m/s$^2$.

EXAMPLE 8

In the process of organizing a colony on the moon, a group of astronauts has constructed a 100 m-high tower. One of them drops a hammer from the top of the tower to the ground.

(a) Use your knowledge of the value of $g$ on the moon (1.6 m/s$^2$) and the definition of acceleration to make a table of the velocity after 1, 2, 3... up to 12 s.

(b) Use the values from your table to make a graph of $v(t)$.

(c) From your knowledge that the displacement is the area under the curve of $v(t)$, construct a table of $x(t)$ for the same 12 s.

(d) From your table, estimate how long it takes for the hammer to fall to the ground.

(e) Estimate its velocity just before it hits the ground.

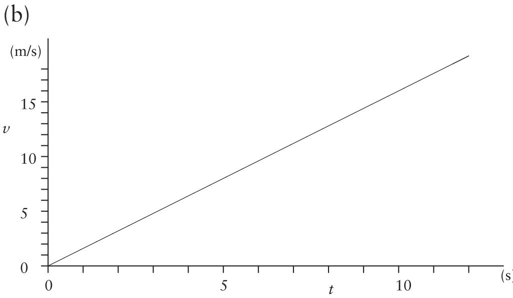
*Ans.:*
We can decide to let either the up or the down direction be positive. The opposite direction is then negative.

Let the downward direction be positive. That avoids having to use a lot of negative numbers while the hammer falls.

(a) The acceleration is the amount by which the velocity changes in each second: here the velocity changes by 1.6 m/s in each second. The velocity starts from zero. It is 1.6 m/s after 1 s, 2×1.6 m/s or 3.2 m/s after 2 s, 4.8 m/s after 3 s, and so on.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $v$ | 0 | 1.6 | 3.2 | 4.8 | 6.4 | 8.0 | 9.6 |
| $t$ | 7 | 8 | 9 | 10 | 11 | 12 | s |
| $v$ | 11.2 | 12.8 | 14.4 | 16.0 | 17.6 | 19.2 | m/s |

(b)




(c) After 1 s the area under the curve of $v(t)$ is that of a triangle with a horizontal side of 1 s and a vertical side of 1.6 m/s, i.e., $(\frac{1}{2})$(1 s)(1.6 m/s), or 1.6 m. After 2 s the area is $(\frac{1}{2})$(2 s)(3.2 m/s), or 3.2 m. After 3 s it is $(\frac{1}{2})$(3 s)(4.8 m/s), or 7.2 m, and so on.

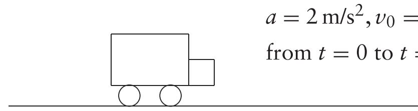
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $x$ | 0 | 1.6 | 3.2 | 7.2 | 12.8 | 20.0 | 28.8 |
| $t$ | 7 | 8 | 9 | 10 | 11 | 12 | s |
| $x$ | 39.2 | 51.2 | 64.8 | 80.0 | 96.8 | 115.2 | m |

(d) From the table we see that after 11 s the hammer has fallen 96.8 m, so that it takes just a little longer to fall the 100 m.

(e) From the graph, after 11 s, $v = 17.6$ m/s. Since it takes a little longer, we can estimate the velocity to be about 18 m/s.

Let's look at the same representations as in the previous section. Here is the verbal statement: a car moves in the $x$ direction with a constant acceleration of 2 m/s$^2$, starting from rest at $t = 0$, and continuing for 10 s.

The picture (or "pictorial representation") is the same as before, but with the information about the motion for the new situation.

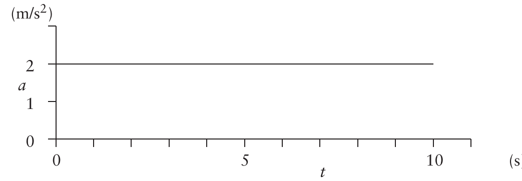


$a = 2$ m/s$^2$, $v_0 = 0$
from $t = 0$ to $t = 10$ s

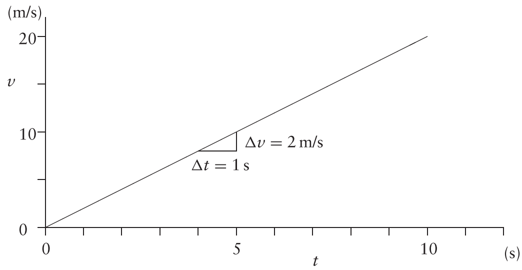
The motion diagram is a series of dots that represent the car's position at different times, separated by equal time intervals. This time the velocity increases, so that the distance between the dots gets larger.



The graph of $a(t)$ is the simplest: $a$ remains the same as $t$ increases. The graph is a horizontal line at the value given for $a$ of 2 m/s$^2$.



The acceleration is the rate at which the velocity changes with time. When the time increases by an amount $\Delta t$, the velocity increases by an amount $\Delta v$. The ratio $\frac{\Delta v}{\Delta t}$ is the acceleration. It is the slope of the graph of $v$ against $t$. Since the car starts from rest, $v = 0$ at $t = 0$.




The figure shows $v(t)$, the relationship between $v$ and $t$. The graph is a straight line, starting with $v = 0$ at $t = 0$, and with a slope of 2 m/s$^2$. After 10 s the velocity is (2 m/s$^2$)(10 s) = 2 m/s.

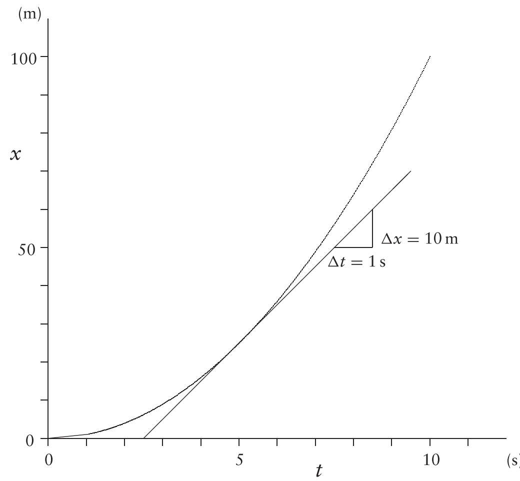
The next step is to go to $x(t)$, the graph of displacement against time. We can use our previous observation that $x$ is given by the area under the curve of $v(t)$. It is therefore the area of the triangle, $(\frac{1}{2})(v)(t)$, which is equal to $(\frac{1}{2})(at)(t)$ or $\frac{1}{2}at^2$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $v = at$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| $x = \frac{1}{2}vt$ | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |

In this case the graph of $x$ against $t$ is not a straight line. (It is a parabola.) As usual, the displacement is measured from a starting point, or, on a graph, from the graph's origin. The *slope*, equal to the velocity, increases as the time increases.

Until now we have talked only about the slope of a straight line. What do we mean by the slope of a curve? To find the slope of a curved line at a point we draw a tangent to the curve, the straight line that just touches the curve at that point. The slope of this straight line is the slope of the curve.

Look, for example, at the point on the curve $x(t)$ where $t$ is equal to 5 s, and draw a tangent there. The value of $x$ at that point is 25 m. Draw

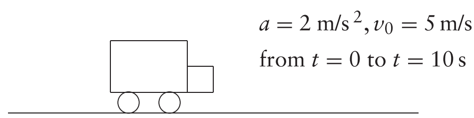



the tangent line at this point. The slope of the tangent is 10 m/s.

We now also have the mathematical representations: $a = 2\frac{m}{s^2}$, $v = at = (2)(t)$ m/s, and $x = \frac{1}{2}at^2$.

We can go one step further by noting that for the same constant acceleration the motion might not start from rest. It could have some *initial* velocity $v_0$ at the time $t = 0$. Let's repeat our example, with $v_0 = 5$ m/s.

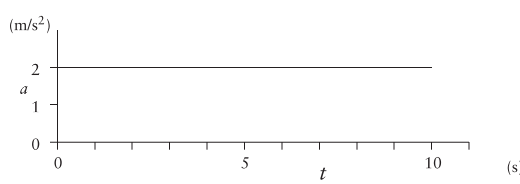
The picture is still the same.



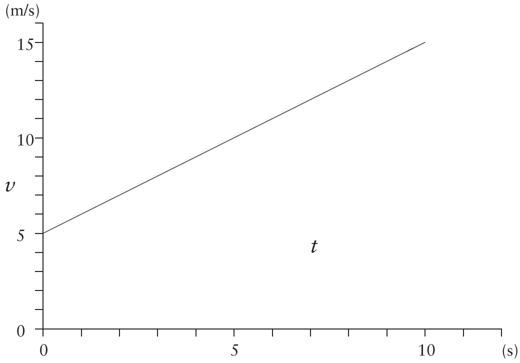

The motion diagram shows the dots starting further apart, and the distance between them continuing to expand as the velocity increases.



The graph of $a(t)$ remains the same, since the acceleration is the same as before.

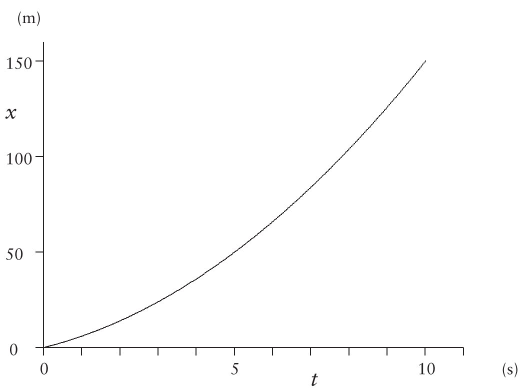


On the graph of $v(t)$ we start at the value $v_0 = 5\,\text{m/s}$ at $t = 0$. Since the acceleration is the same as before, the graph is still a straight line with a slope of $a = 2\,\text{m/s}^2$. Its mathematical representation is $v = v_0 + at$.




We can again compare this expression with the standard expression for a straight line, $y = mx + b$. This time we plot $v$ along the vertical axis and $t$ along the horizontal. The slope is $a$ and the intercept on the vertical axis is $v_0$. The mathematical expression becomes $v = v_0 + at$.

To get the displacement $x(t)$ we look at the area under the curve. This time it is the sum of the rectangle whose area is $v_0 t$ and the triangle that we had before, whose area is $\frac{1}{2}at^2$, giving $x = v_0 t + \frac{1}{2}at^2$. (The shape is again that of a parabola, but the slope at the origin is no longer zero.)




We have gone a long way toward solving the problem that we set ourselves at the beginning of the chapter for this special case. It was to take a particle at some starting point and starting time, with a known initial velocity at that place and time, to figure out where it would be at some later time, and how fast it would then be going.

Let our particle start out at $x = 0, t = 0$, with velocity $v_0$. The two mathematical statements,

$$x = v_0 t + \frac{1}{2}at^2 \tag{3.1}$$

and

$$v = v_0 + at \tag{3.2}$$

then do just what we were looking for. They tell us what $x$ and $v$ will be at any time, $t$. All we have to know is that the acceleration is constant, and what its value is.

That isn't quite what we promised. We talked about forces, not acceleration. But there is a close relationship between force and acceleration, which we will explore in the next chapter.

In the meantime these two relations describe what happens when a particle moves along a straight line with constant acceleration. It is often convenient to use a relation that does not contain the time explicitly. We can get this third relation by eliminating the time from the first two.

To do this, use the second relation to show what $t$ is equal to, namely $t = \frac{v - v_0}{a}$, and substitute the right-hand side of this expression for $t$ in the first relation: $x = (v_0)(\frac{v-v_0}{a}) + (\frac{1}{2})(a)(\frac{v-v_0}{a})^2$. If you now multiply every term by $2a$, you get $2ax = 2v_0(v - v_0) + (v - v_0)^2 = 2v_0 v - 2v_0^2 + v^2 - 2vv_0 + v_0^2 = v^2 - v_0^2$, which gives our third relation for constant acceleration:

$$v^2 = v_0^2 + 2ax \tag{3.3}$$

Equations (3.1), (3.2), and (3.3) are three relations that give a complete kinematic description of motion in a straight line with constant acceleration.

EXAMPLE 9

Go to the PhET website (http://phet.colorado.edu) and open the simulation *The Moving Man*. (See the earlier example on this simulation.) Go to *Charts*.

(a) Put in $x = 0$, $v = 0$, and $a = 0.4\,\text{m/s}^2$.

What is the shape of $v(t)$? What is its equation (up to where he hits the wall)?

What is the shape of $x(t)$? What is its equation? Check it with the Expression Evaluator.

(b) Put in $x = -10$ m, $v = 5$ m/s, $a = -1$ m/s$^2$.
Watch the velocity vector.
Watch how $x(t)$ follows the man's position.
What does the graph of $v(t)$ show? What is its equation? Check that the equation gives the values of $v$ at $t = 0, 5$ s, and 10 s. What is the velocity at the point where the man turns around?
What does $a(t)$ show? What is its equation?

(c) What is the equation of $x(t)$? What is the value of $x$ where the man turns around?

(d) Check your expressions with the Expression Evaluator.

*Ans.:*

(a) $v(t)$ is a straight line. Its slope is 0.4 m/s, which is equal to the acceleration. $v = 0.4t$ $(v = at)$. $x(t)$ is a parabola: $x = 0.2t^2$ $(x = \frac{1}{2}at^2)$.

(b) While the man moves to the right (in the positive direction), $v$ is positive. It is zero where he turns around, and then negative while he moves to the left. $x(t)$ starts at $x = -10$ m/s and goes past zero and then back, decreasing toward $-10$ where the man hits a wall.
$v(t)$ is a straight line with the negative slope $-1$ m/s$^2$. Its equation is $v = 5 - t$ $(v = v_0 + at)$. The graph and the equation show that at $t = 0$, $v = 5$ m/s, at $t = 5$ s, $v = 0$, and at $t = 10$ s, $v = -5$ m/s. At the point where the man turns around, $v = 0$.
$a$ is constant at $-1$ m/s$^2$.

(c) The equation for $x$ is more difficult to determine, because it does not follow the relation in the text, which is for a particle that starts out at $x = 0$. Here it starts at $x = -10$ m and the $-10$ m has to be added to the expression $x = v_0t + \frac{1}{2}at^2$ to give $x = -10 + 5t - \frac{1}{2}t^2$.
At $t = 0$, $x = -10$ m, at $t = 10$ s, $x = -10 + 50 - 50 = -10$ m, and at $t = 5$ s when the man turns around, it is $-10 = 25 - 12.5$ or $x = 2.5$ m, as seen on the graph.

---

We have dealt with just two special cases of kinematics: first motion along a straight line with constant velocity and then motion along a straight line with constant acceleration. We could now examine motion with different kinds of accelerations, as in oscillatory motion, motion in a plane (two-dimensional motion), such as projectile motion and circular motion, and motion in space (three-dimensional motion), as in a spiral. The definitions of acceleration, velocity, and displacement remain the same, but the relation between these quantities is different for each kind of motion. Each fulfills the promise of a totally determined motion once the position and velocity at a given moment (the *initial conditions*) and the acceleration, $a(t)$, are given. In each case there are different characteristic features.

But no matter how complicated the calculations for a particular case are, the content of the calculations of kinematics is still the same: displacement, velocity, acceleration, time, and the relationships between them as they follow from the definitions. There are only the definitions: no other laws or principles are required. Once the acceleration $a(t)$ is known, there is no further recourse to nature or to observation. The definitions provide us with all we need to describe the motion.

EXAMPLE 10

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A bicyclist, riding with a velocity of 4 m/s, slows down at a steady rate of 1.5 m/s$^2$ until she stops.

(a) Draw a graph of $v(t)$ and use it to determine how far she moves until she stops and how long it takes her to stop.

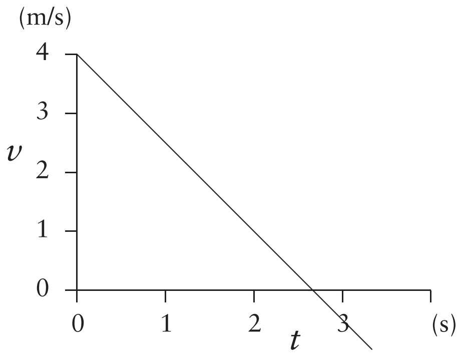
(b) Write a mathematical statement describing her displacement $x(t)$ while she is slowing down.

(c) Write a mathematical statement for $v(t)$ while she is slowing down.

(d) Use your mathematical statements to find the time and distance until she stops and compare them to your answers in part (a).

*Ans.:*

(a) Let's take the direction of the starting velocity to be positive. If $v$ increases, $a$ is then positive. Here it decreases, and is therefore negative: $a = -1.5$ m/s$^2$.
After 1 s the velocity has decreased by 1.5 m/s, from 4 m/s to $4 - 1.5$ or 2.5 m/s. After 2 s it has decreased by another 1.5 m/s, to 1 m/s.




The line crosses the horizontal axis where the velocity has decreased to zero. That is after about 2.7 s.

The distance she has covered by that time is equal to the area under the curve, i.e., the area of the triangle whose sides are 4 m/s and 2.7 m. This area is $(\frac{1}{2})(4\,\text{m/s})(2.7\,\text{s}) = 5.4\,\text{m}$.

(b) $x = v_0 t + \frac{1}{2}at^2$, where $v_0 = 4\,\text{m/s}$ and $a = -1.5\,\text{m/s}^2$, so that $x = 4t - 0.75t^2$.

(c) $v = v_0 + at$, so that $v = 4 - 1.5t$.

(d) From the statement in part (c), $v = 0$ when $0 = 4 - 1.5t$, or $1.5t = 4$, or $t = 4/1.5 = 2.67$. When $t = 2.67$, the answer to part (b) shows that $x = (4)(2.67) - (.75)(2.67^2) = 10.67 - 5.33 = 5.33\,\text{m}$. (This answer is more precise than that for [a], which was just read off the graph.)

EXAMPLE 11

A ball is thrown upward with an initial velocity whose magnitude is 20 m/s. It moves upward and then returns to the place where it started. We will assume that air resistance may be neglected, so that the magnitude of the downward acceleration is $g = 9.8\,\text{m/s}^2$.

(a) Develop the mathematical statements for the displacement $y(t)$ and the velocity $v(t)$.

(b) Draw graphs of $a(t), v(t)$, and $y(t)$.

(c) What is the velocity when the ball returns to the starting point?

(d) What is the maximum height to which the ball rises?

(e) What is the length of time during which the ball is in the air?

*Ans.:*

We have to decide which direction we will take as positive. (It is up to us to decide, but we have to stick to the decision for the duration of the problem.) Here we will take the upward direction to be positive.

(a) We can use the first two relations for constant acceleration, $x = v_0 t + \frac{1}{2}at^2$ and $v = v_0 + at$, and use $y$ instead of $x$, with $g = -9.8\,\text{m/s}^2$ (negative, because we have chosen up to be positive, and the acceleration is down) and $v_0 = 20\,\text{m/s}$, to get $y = 20t - 4.9t^2$ and $v = 20 - 9.8t$.

(c) There are different ways to get the answer. The simplest is to use the third of the equations that describe constant acceleration, $v^2 = v_0^2 + 2ax$. The displacement for the entire motion is now zero, so that we are left with $v^2 = v_0^2$, which has the solutions $v = \pm v_0$. Since the direction at the starting point is up, and therefore positive, it is negative on the way down, and we need to choose the negative value, $v = -v_0, = -20\,\text{m/s}$.

We can also begin by answering the next two parts of the question (d and e): at the top $v = 0$. We can use $v = v_0 + at$ to find the time to reach the top: $0 = 20 - 9.8t$, which yields $t = \frac{20}{9.8} = 2.04\,\text{s}$. We can then find $y$ from $y = 20t - 4.9t^2 = 40.8 - 20.4 = 20.4\,\text{m}$.

The motion downward can be analyzed in the same way: now the initial velocity (starting from the top) is zero, and the displacement to the starting point is $-20.4\,\text{m}$ (down, and therefore negative). We can use $v^2 = v_0^2 + 2ax$, now $v^2 = 0 + (2)(-9.8)(-20.4) = 400$, so that $v = \pm 20\,\text{m/s}$. We know that the direction is negative, so that we have to choose the negative sign, as before, $v = -20\,\text{m/s}$.

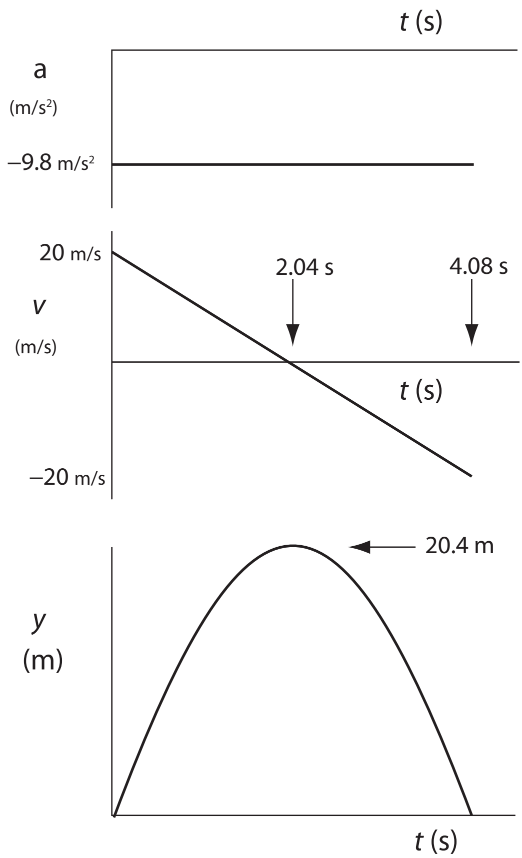
We can find $t$ from $v = v_0 + at$, or $t = \frac{v - v_0}{a} = \frac{-20-0}{-9.8} = 2.04\,\text{s}$, as before.

(d) The graphs of part (b) are well worth contemplating.

$a(t)$ is constant and negative with magnitude $g = 9.8\,\text{m/s}^2$. This is also the slope of the graph of $v(t)$.

The graph of $v(t)$ is positive for half of the flight of the ball, until it reaches its highest point. During this part of the flight the velocity points up. It starts from 20 m/s and decreases, reaching zero when the ball gets to its highest point. During the second half of the flight the velocity is negative, as the ball falls back down to the starting point, where it has the same speed as initially, but in the opposite direction, so that the velocity has the same magnitude, but opposite sign.

$y(t)$ starts and ends at $y = 0$. Half way between these two points the ball is at its highest point and $y$ is at its maximum value. The slope of $y(t)$ starts out positive and decreases during the first half until it is zero when the ball is at its maximum height. It is then negative, increasing in magnitude as the ball returns to its initial position. The slope of $y(t)$ is equal to $v(t)$.




## 3.3 The mathematics of change

There is a branch of mathematics that deals with two of the concepts that we have just talked about: the slope of a curve and the area under a curve. It describes the relationships between changing quantities, i.e., how changes in one quantity affect another. It was developed to a large extent in the seventeenth century by Isaac Newton, Gottfried Wilhelm Leibniz, and others, and the primary motivation was to describe motion. We won't study this subject in detail, but we will look at some of its concepts.

We already know the relation between velocity and displacement and between acceleration and velocity. There are many other relationships between varying quantities, involving rates of change, and changes in the rate of change. They might deal with other mechanical quantities, or perhaps with the rate of transformation of electrical energy to internal energy. Other examples in which rates of change are important might be the way in which labor costs affect profits, or how the number of predators and the food supply affect animal populations.

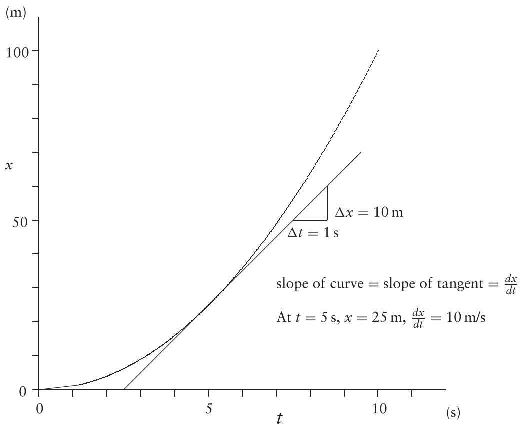
The subject (the *calculus*) opens up a fascinating and wonderful world. We will look at some of its features and symbols, primarily to enhance and clarify topics that you already know.

### *Slopes and derivatives*

Earlier we used the fact that the slope of the line of $x$ plotted against $t$ is $\frac{\Delta x}{\Delta t}$, the change of the displacement ($\Delta x$), divided by the time interval ($\Delta t$) during which the displacement changes by $\Delta x$. That was for a straight–line relationship between $x$ and $t$. For a relationship represented by a curved line it was necessary to draw the tangent to the curve, and to use the changes in $x$ and $t$, namely $\Delta x$ and $\Delta t$, for the tangent line. We now introduce a shorthand notation for the slope of the tangent, namely $\frac{dx}{dt}$.

The quantity $\frac{dx}{dt}$ is the slope of the tangent line. It is also called "the derivative of $x$ with respect to $t$." Instead of saying "the velocity is the slope of the graph of displacement against time," we can say that "the velocity is the derivative of the displacement with respect to time." The words may be unfamiliar, but what we mean to say is exactly the same. We can now use $v = \frac{dx}{dt}$ as the definition of the velocity.

We haven't done anything. We have only introduced a new word for "slope," namely "derivative," and a new symbol, $\frac{dx}{dt}$. Similarly, we can define the acceleration to be $a = \frac{dv}{dt}$.




Of course this isn't the whole story. The methods of the calculus also tell us how to calculate derivatives. Here is an example of the kind of result that we can get: if the relationship between $x$ and $t$ is given by $x = t^2$, then the slope $\frac{dx}{dt}$ is equal to $2t$.

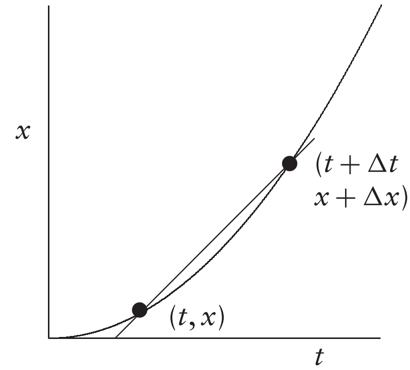
---

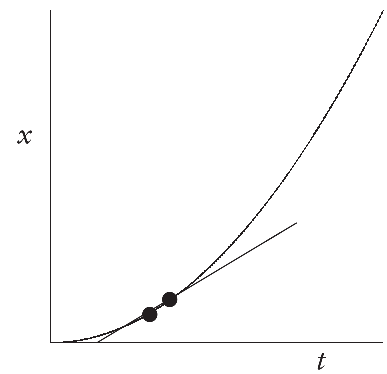
We can see how this comes about. Look at two points on the graph of $x(t)$, namely $(t, x)$, and a point not too far away, $(t + \Delta t, x + \Delta x)$. Since both points are on the curve $x = t^2$, we can write $x = t^2$ and $(x + \Delta x) = (t + \Delta t)^2 = t^2 + 2t\Delta t + (\Delta t)^2$. Subtracting the first of these from the second, we get $\Delta x = 2t\Delta t + (\Delta t)^2$ or $\frac{\Delta x}{\Delta t} = 2t + \Delta t$. Now let the second point move toward the first. As $\Delta t$ becomes smaller and smaller, the line through the two points becomes the tangent at $(t, x)$, and $\frac{\Delta x}{\Delta t}$ becomes $\frac{dx}{dt}$, equal to $2t$.

More generally, for $x = A\,t^n$, where $A$ and $n$ are fixed numbers, it can be shown that the derivative $\frac{dx}{dt}$, the slope of the curve of $x$ against $t$, is equal to $nAt^{n-1}$. [For $x = t^2$, we have $v = \frac{dx}{dt} = (2)(t^{2-1}) = 2t$.]

Let's see what this leads to for our old case of constant acceleration. Start with $x = v_0 t + \frac{1}{2}at^2$, and find the expressions for $v$ and $a$.

We'll have to take one term at a time. For the term $v_0 t$, $v_0$ is a number, taking the place of $A$. The exponent $n$ is one ($t^1 = t$). The derivative therefore becomes $(1)(v_0)(t^0)$. Since $t^0 = 1$, this is just $v_0$. Now look at the second term, $\frac{1}{2}at^2$. This time the exponent $n$ is two, and the constant term (corresponding to $A$) is $\frac{1}{2}a$. The derivative is $(2)(\frac{1}{2}a)(t^1)$, or $at$. Taking the two terms together we see that $\frac{dx}{dt} = v_0 + at$. In other words, $v = v_0 + at$, just as we had it earlier.



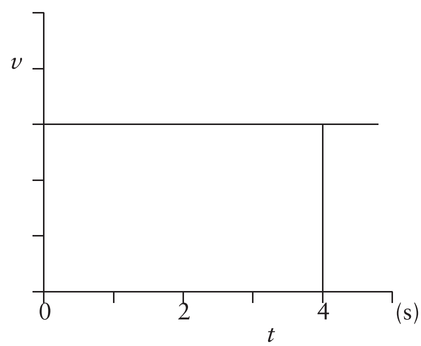

We can apply the same procedure to $v = v_0 + at$, to find the derivative $\frac{dv}{dt}$, i.e., the slope of the graph of $v$ against $t$, which we know to be the acceleration. The first term is $v_0$, which is constant, so that there is no change, and therefore no derivative. From the second term (where $n = 1$) we get $(1)(a)(t^0)$, or $a$, the constant value of the acceleration.

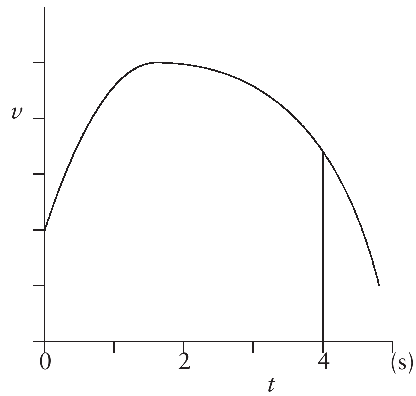
The particular relations between $x$, $v$, and $a$ that we have used apply only when $a$ is constant. But the definitions $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$ are always valid.

---

## *Areas and integrals*

Now let's look at the area under the curve of $v$ against $t$. In the simplest case it can be a rectangle, but it can also have a more complicated shape. Consider two examples in which we look at the area between the values $t = 0$ and $t = 4$ s. Here is the calculus terminology for that: this same area is called "the integral of $v$ with respect to $t$ between the limits $t = 0$ and $t = 4$ s." The symbol for it is $\int_0^{4s} v\, dt$. This turns out to be the opposite of taking the derivative, and is sometimes called the antiderivative.




Again, we haven't done anything, except to use different symbols. If you are not familiar with them they look strange, perhaps forbidding. But then you realize that they are just symbols, just new ways to write down something that you already know.

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For the particular case where $v = At^n$, $\int v\,dt = A\frac{t^{n+1}}{n+1}$. You can see that going backward, taking the derivative, gives you $(n+1)(A)\frac{t^{n+1-1}}{n+1}$, or $At^n$.

---

You may not have had a course in calculus, but here you see its basic ideas and will be able to use them. We won't do any calculations using calculus in this book, but we will use the ideas and the symbols occasionally.

Go to the PhET website (http://phet.colorado.edu) and open the simulation *Calculus Grapher*.

It allows you to explore derivatives and integrals. Start by checking "derivative" and choosing the triangle. Use the mouse to drag the *f*(*x*) line up from the *x*-axis. The vertical scale can be changed with the buttons on the left and the horizontal scale with the slider under the top figure on the right. Each time you want a different horizontal scale or to go to a new function, press zero and start again.

Try the various other functions. In each case see that the derivative graph ($\frac{df}{dx}$) shows the slope of the graph of *f*(*x*). Pay particular attention to the sinusoidal function (third down on the left) to which we will come back later: get *f*(*x*) to start upward at the origin so that it represents the sine function and continues for about one complete oscillation (up, down, and back). The derivative then represents the cosine function, starting at its maximum. What is the slope when *f*(*x*) is zero and going up, at its maximum, at its minimum, and at zero going down?

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EXAMPLE 12

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The astronaut revisited.

(a) Write mathematical statements for the astronaut's hammer in Example 8 for $v(t)$ and $x(t)$.

(b) Use your statement to find the answers to parts (d) and (e) of Example 8, and compare them to your previous answers.

*Ans.:*

(a) This time assume that upward directions are positive. $v = at$, where $a = -1.6\,\text{m/s}^2$, so that $v = -1.6t$.

$$x = \tfrac{1}{2}at^2, \text{ or } x = -0.8t^2.$$

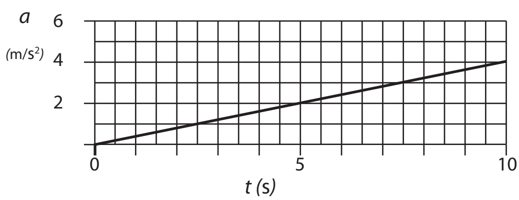
(b) Use the second relation first: $x = -100$ m where we continue to use upward as positive so that the downward direction is negative. $-100 = -0.8t^2$, or $t^2 = \frac{100}{.8} = 125$, so that $t = 11.2$ s. (Note that, as usual, we have used SI units for all the data, so that the answer automatically comes out in the appropriate SI unit also.)

$v = at = (-1.5)(11.2) = -16.8$ m/s. The velocity just before the hammer hits the ground has the magnitude 16.8 m/s, and is in the negative or downward direction.

(We could have chosen the downward direction to be positive. All downward-directed vectors would then be positive. The calculation would look different, but the result would be the same.)

EXAMPLE 13

The graph shows the acceleration of an object.




(a) Write down a mathematical statement that describes how the acceleration changes with time.

(b) Find $v(t)$.

(c) Previous examples have been for constant velocity or constant acceleration. How would you describe the motion of this graph? Explain.

*Ans.:*

(a) $a = 0.4t$.

(b) The velocity is the area under the curve: After 1 s it is $(\frac{1}{2})(1)(0.4)$. After 2 s it is $(\frac{1}{2})(2)(0.8)$. After $t$ s it is $(\frac{1}{2})(t)(0.4t)$ or $0.2t^2$.
Note that if $v = 0.2t^2, a = \frac{dv}{dt} = 0.4t$.

(c) Neither $v$ nor $a$ is constant here. The acceleration increases linearly with time, i.e., it is proportional to $t$.

## 3.4 Summary

Everything around us is in motion. Even an object that appears to be at rest consists of atoms in motion, with their electrons and their nuclei also in motion. To describe an object in motion we use a number of concepts, primarily *displacement, velocity*, and *acceleration*. The displacement shows how far the object is from the start or from a reference point. The velocity describes how fast it moves. The acceleration measures how fast the velocity changes.

Most objects or systems are too complex to be analyzed completely. We therefore invent a *model*, a simplified version of the real system. It retains some of the essential features and allows an approximate analysis by neglecting some of the complicating aspects. For example, in this and the following chapters we often treat an object as a particle.

The displacement is the distance to a point, from a starting point or from a reference point. The velocity is the rate at which the displacement changes. If the displacement is in meters, we can measure the velocity in meters per second, m/s. If the displacement changes by $\Delta x$ meters in a time $\Delta t$ s, the velocity is $v = \frac{\Delta x}{\Delta t}$. On a graph of $x$ against $t$ this is the slope.

The displacement, the velocity, and the acceleration are *vector* quantities. They have both magnitude and direction. A vector can be represented by a line whose length indicates the magnitude, with the direction indicated by an arrow. The magnitude of the velocity (represented by the length of the velocity vector) is the *speed*. If the motion is along a straight line we need only positive and negative values for $x, v$, and $a$ to express the magnitude and the direction.

If a graph of $x$ against $t$ is curved, the *slope* at a point is the slope of the line that is tangent to the curve at that point. It is also called the *derivative* of $x$ with respect to $t$, and is written $\frac{dx}{dt}$. The definitions can be written as $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$. Since $a$ is the derivative with respect to $t$ of the derivative of $x$ with respect to $t$, it is also called the *second derivative* of $x$ with respect to $t$, and is written $\frac{d^2x}{dt^2}$. For $x = At^n$, $\frac{dx}{dt} = nAt^{n-1}$.

We go from $x$ to $v$ and to $a$ by taking derivatives. We can go in the opposite direction with areas under the curves or *integrals*.

## 3.5 Review activities and problems

### *Guided review*

1. Cindy bicycles to the grocery store. She first goes four blocks west and then three blocks south. Find the magnitude and direction of her displacement.

2. Cindy bicycles along the path described in Question 1. It takes her 5 minutes for the first part (going west) and 4 minutes for the second part (going south).

(a) What is her average velocity in each of these two parts? Assume that each block is 150 m and express each of the two velocities in m/s.

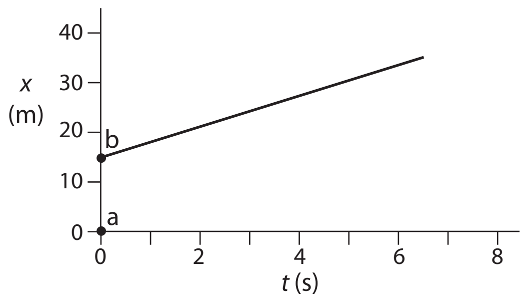
(b) Draw a graph of $x(t)$, using meters and seconds, for the first part of her path.

3. Draw motion diagrams for a car that is

(a) accelerating with a positive acceleration to the right,

(b) slowing down as it moves to the right.

4. The graph shows Cindy again as she bicycles along a straight path.




(a) How far does she move in 5 s?

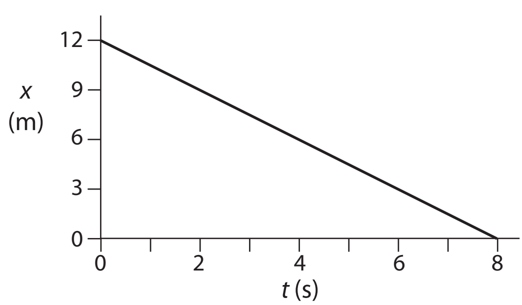
(b) What do points *a* and *b* on the graph represent about Cindy's path?

(c) What is the slope of the line?

(d) What is her velocity?

(e) Write the equation for the line, first using only symbols, and then using only numbers except for $x$ and $t$.

5. The graph shows Harry's motion. $x$ is positive to the right.




(a) Where does Harry's motion begin?

(b) At what time does he reach the point represented by $x = 0$?

(c) In what direction does he move?

(d) What are the magnitude and direction of his velocity? Does the direction of the velocity change? How can you tell? How can you tell whether his velocity is constant?

6. Go to the PhET website and open the simulation *The Moving Man*. Put in $x = 0$ and $v = -2$ m/s.

What does the graph of $v(t)$ show?

What happens to it when the man gets to the wall?

What is the equation for $v(t)$ before he gets to the wall?

What does the graph of $x(t)$ show?

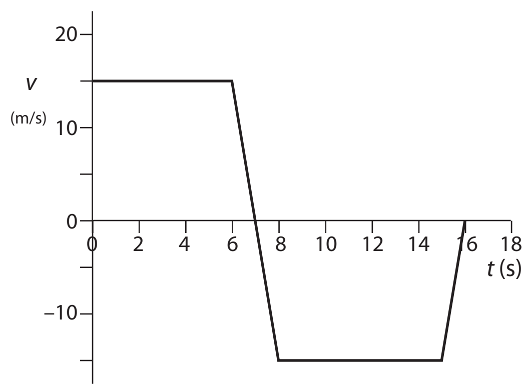
What happens to it when the man gets to the wall?

What is the equation for $x(t)$ before he gets to the wall?

What does the graph of $a(t)$ show?

Put your expression for $x(t)$ into the Expression Evaluator and compare what you see to the "experimental" graph.

7. The graph shows the motion of your car for 16 s. The positive direction is to the right.




What is the type of motion of the car between

(a) 0 and 6 s

(b) 8 and 15 s

(c) 6 and 8 s. What happens at $t = 7$ s?

(d) 15 and 16 s

8. Tehmina's car accelerates from rest with $a = 5\ \text{m/s}^2$.

(a) Make a table of $v$ and $t$ at 1 s intervals for 10 s.

(b) Make a graph of $v(t)$ for the 10 s.

(c) Make a table of $x$ and $t$ for the 10 s. (Let $x = 0$ when $t = 0$.)

(d) How long does it take for the car to reach a velocity of 40 km/h? How far has it then traveled?

9. Go to the PhET website and open the simulation *The Moving Man*.

Put in $x = 10$, $v = -5$, and $a = 1$.

(a) Watch the velocity vector and how the graph of $x(t)$ follows the man's position.

What does the graph of $v(t)$ show? What is its equation? Check that the equation gives the values on the graph. What is the velocity when the man turns around? Check your equation with the Expression Evaluator.

What does $a(t)$ show? What is its equation?

(c) What is the equation of $x(t)$? What is the position where the man turns around? Check your equation with the Expression Evaluator.

10. Tony, riding a motorcycle, with a velocity of 21 m/s slows down with an acceleration whose magnitude is 3 m/s$^2$.

(a) Draw a graph of $v(t)$ to the time when the motorcycle stops.

(b) Use your graph to determine how far he moves before the motorcycle stops and how much time that takes.

(c) Write mathematical statements for $x(t)$ and $v(t)$. Use your mathematical statements to find the answers to part (b), and compare them to the values that you obtained graphically.

11. A stone is thrown vertically upward from outside a window, with $v_0 = 18$ m/s. It moves up, and then moves down to the ground 420 m below the point where it started. Let "up" be positive.

(a) Write down the mathematical statements for $a(t), v(t)$, and $y(t)$.

(b) Draw graphs of these functions.

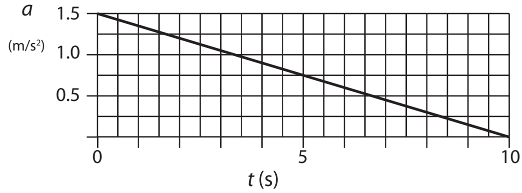
(c) What is the maximum height above the window to which the stone rises?

(d) How long is the rock in the air before it reaches the ground?

(e) What is its velocity just before it hits the ground?

12. (a) Write down the mathematical statements for $v(t)$ and $x(t)$ of the car in Question 8 of this Guided Review.

(b) Use your statement to find how long it takes for the car to reach 40 miles per hour and how far it has then traveled. Compare your answers to those of Question 8.



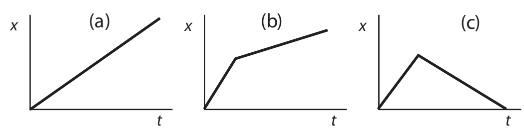

13. (a) Describe a procedure for finding $v(t)$ graphically for the $a(t)$ shown in the figure.

(b) Find $v(t)$.

(c) Check to see what $v(t)$ is by using the relation for the integral of $a(t)$ and compare the result to your answer to part (b).

## *Problems and reasoning skill building*

1. The figure shows three qualitative position-time graphs.




For each one

(a) draw a qualitative velocity-time graph,

(b) describe in words what happens, i.e., state what changes occur in the velocity and in the direction of motion.

(c) In the second and third graphs there is a kink in $x(t)$. What happens to the velocity at the kink? What is the acceleration at the kink? What is your comment on how realistic this is?

2. Draw qualitative graphs of $x(t)$ and $v(t)$ for an object that moves with constant speed away from the origin, stops briefly, and then continues in the same direction as before at twice the original speed. It stops again briefly, and then moves toward the origin at the original speed until it returns to the starting position.

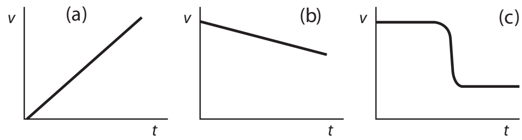
3. A car moves toward the origin at 20 m/s for 5 s, stops, and then (nearly instantly) moves away from the origin at 20 m/s for the next 5 s.

(a) Draw a graph of $v(t)$.

(b) What is the change in the speed at $t = 5$ s?

(c) What is the change in the velocity at $t =$ 5 s?

4. The following are qualitative graphs of $v(t)$. For each one sketch the corresponding graph of $a(t)$.




5. Sketch qualitative graphs of $x(t)$, $v(t)$, and $a(t)$ for a car that moves

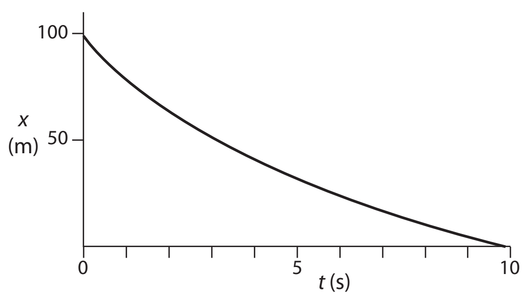
(a) away from the origin, in the positive direction, with constant speed,

(b) toward the origin with constant speed,

(c) away from the origin, starting from rest, with a constant positive acceleration,

(d) past the origin with a positive velocity, with a constant negative acceleration until it stops.

6. Describe the motion corresponding to the graph. Where does the object start and end? What are the magnitude and direction of the initial velocity, and how does the velocity change? What is the sign of the acceleration?



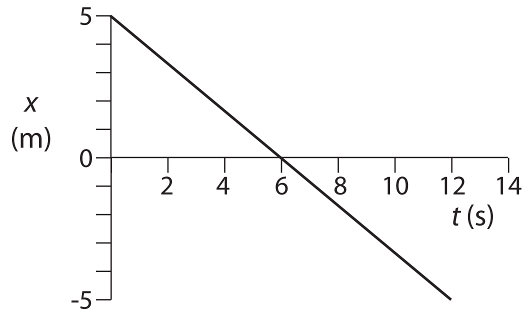

7. Two football players run toward each other with constant velocities, Bill to the right with twice the speed of Tom, who runs to the left, starting 25 m from Bill.

(a) Draw a motion diagram for the two players.

(b) Draw graphs of $v(t)$ for both on the same coordinate system.

(c) Draw graphs of $x(t)$ for both on the same coordinate system.

8. The graph shows $x(t)$ for a marble. The positive direction is to the right.



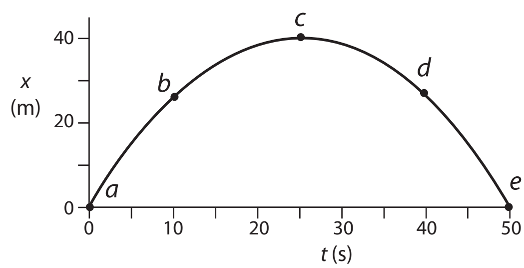

(a) What is the magnitude and direction of the velocity?

(b) What is the acceleration?

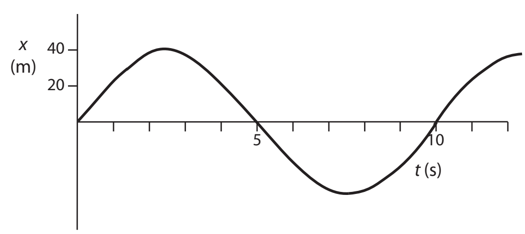
(c) How far does the marble travel?

(d) What is its average speed?

9. For the $x(t)$ of the diagram, what are the approximate values of $v(t)$ at points $a$, $b$, $c$, $d$, and $e$? Sketch a diagram of $v(t)$.



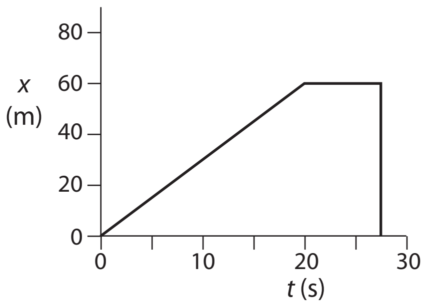

10. For the $x(t)$ of the diagram, what are the approximate values of $v(t)$ at 1, 2, 3, . . . to 12 s? Sketch the corresponding diagram of $v(t)$.

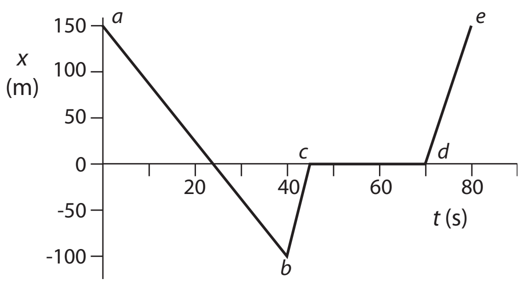



11. Draw a graph of $x(t)$ for a bicyclist who rides at 5 m/s to the right for 15 s, stops for 10 s, and then turns around to go back to the starting point, which takes her 25 s more.

12. Describe (in words) the motion represented by the following graph of $x(t)$:

13. Describe (in words) the motion represented by the following graph of $x(t)$:




14. For the diagram of the previous question, what is the average speed between

(a) *a* and *b*

(b) *a* and *c*

(c) *a* and *d*

(d) *a* and *e*

15. A molecule travels to a wall with a constant speed of 100 m/s. It bounces back, and then travels in the opposite direction with a speed of 90 m/s.

Describe what happens at the wall to its speed, its velocity, and its acceleration.

16. You run half-way around a circular track whose radius is 60 m at a steady 5 m/s. What are the following for this part of your path:

(a) displacement

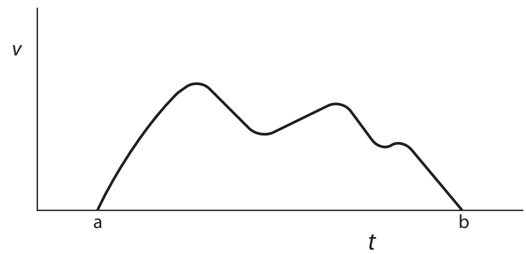
(b) average speed

(c) change in the velocity

17. The figure shows a graph of $v(t)$. How can you find the average velocity between points *a* and *b* from this graph?

18. You stand at the edge of a cliff and throw a rock straight up with an initial velocity of 15 m/s. It reaches a maximum height, $y_{max}$, above the starting point, and then falls past you to a point 60 m below where it started.

(a) What is the maximum height of the rock from the starting point?




(b) How much time has elapsed to the time when the rock passes its starting point on the way down?

(c) What is its speed just before it hits the ground 60 m below you?

(d) What is the total time that the rock is in the air?

19. What are the displacement and the velocity of the rock of the previous question when the elapsed time is

(a) 1 s

(b) 2 s

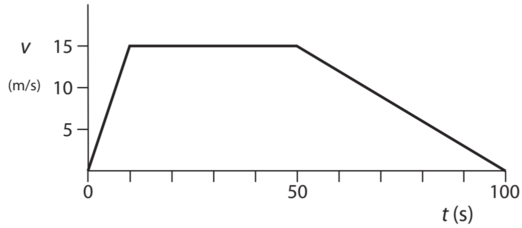
(c) 5 s?

20. (a) My house is 16 miles from my office. On Monday it took me 40 min to get from my house to the office. What was the magnitude of my average velocity in miles per hour?

(b) On Tuesday I traveled the same route at 40 miles per hour for 20 minutes, and at 30 miles per hour for 15 minutes. I was stopped for 5 minutes at stoplights. What was the magnitude of my average velocity in miles per hour?

(c) What is missing in the description of part (b) so that it is unrealistic?

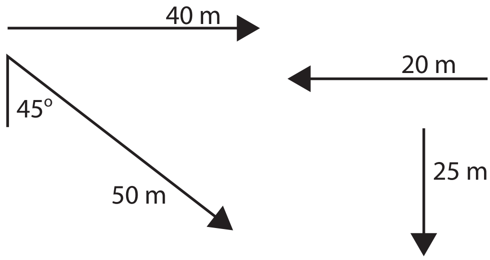
21. The figure shows a graph of $v(t)$. What is the average velocity for the trip?




22. A car moves in a straight line at 40 m/s for 20 s and then at 30 m/s for 30 s. What is the average velocity?

23. What is the sum of the following displacements? First find the magnitude and direction

graphically. Then calculate the magnitude from the vector diagram.




24. The following problems are about a ball moving straight up or down. We will assume that air resistance may be neglected, so that the acceleration is downward and has a magnitude of 9.8 m/s$^2$.

(a) The ball starts from rest and falls for 5 s. State your decision as to whether you wish up or down to be positive. Write down the expressions for $v(t)$ and $x(t)$ using numbers for every quantity except $t$. Draw graphs of $y(t)$ and $v(t)$. What are $v$ and $y$ when $t = 5$ s, from your graph and analytically?

(b) Answer the questions of part (a) for a ball that is thrown downward with an initial speed of $5 \frac{m}{s}$.

(c) Answer the questions of part (a) for a ball with an initial velocity of 5 m/s upward.

(d) Write down expressions for $v(x)$ for parts (a), (b), and (c).

25. A squirrel running at 3 m/s sees that it is pursued by a cat and accelerates to 8 m/s in 2 s. It travels 14 m in that time.

(a) What is its average acceleration?

(b) Is the acceleration constant? How can you tell?

26. As you drive along a highway at 50 miles per hour you suddenly see a deer coming into your path. Your reaction time is 0.25 s and your acceleration has a magnitude of 14 m/s$^2$. How far do you travel before your car stops?

27. (a) What is the average speed, in miles per hour, of a world class sprinter who runs 100 m in 10 s?

(b) What is the average speed (in miles per hour) of a runner who runs a mile in 4 minutes?

(c) What percentage is this of the answer to part (a)?

28. You are driving at 65 miles per hour, following a car 20 feet ahead of you, which is moving at the same speed. The car ahead of you crashes into a car that has drifted into its lane and comes to rest almost instantaneously. Your reaction time is 0.3 s.

(a) Describe your subsequent motion.

(b) If you step hard on your brakes, so that your acceleration has a magnitude of 15 m/s$^2$, how far behind the other car (when it crashes) do you need to be to avoid a collision?

(c) You have had two beers, and your reaction time has doubled. What is the answer to part (b) now?

## *Multiple choice questions*

1. A ball is thrown upward from $y = 0$ and then falls back to its starting point. Neglect air resistance and let up be positive. Which of the following changes sign during the flight?

(a) $y$
(b) $v$
(c) $a$
(d) none of the above

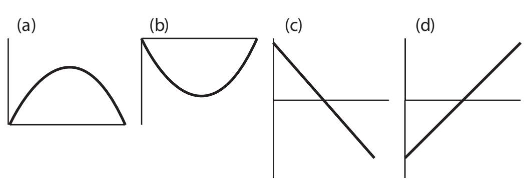
2. For the ball of the previous question, which of the following is zero at the highest point?

(a) $v$
(b) $a$
(c) $v$ and $a$
(d) $y, v$, and $a$

3. For the ball of Question 1, which of the following changes sign at the highest point?

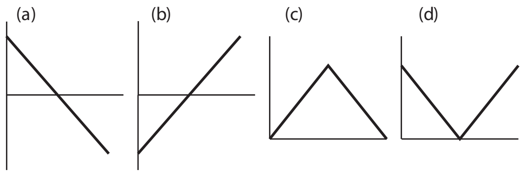
(a) $x$
(b) $v$
(c) $x$ and $v$
(d) $x, v$, and $a$

4. Which among the following pairs of graphs are corresponding graphs of $x(t)$ and $v(t)$?

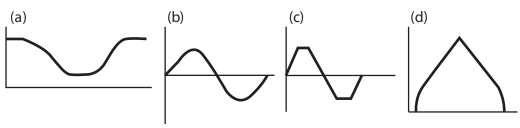



(a) $a$ and $d$ and $b$ and $c$
(b) $a$ and $c$ and $b$ and $d$
(c) $a$ and $d$ only
(d) $b$ and $d$ only

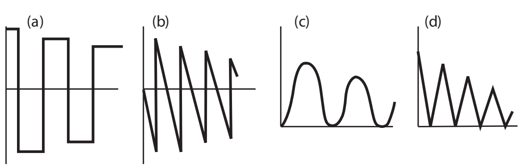
5. A toy car moves to the left with constant speed, then turns around and moves to the right with constant speed until it reaches its starting point. Which of the diagrams of $x(t)$ represents this motion? (Positive is to the right.)




6. A toy car starts from rest, speeds up gradually to a constant speed, and slows down until it comes to rest. It speeds up gradually in the opposite direction, slows down, and comes to rest at its starting point. Which of the following graphs of $x(t)$ represents this motion?




7. Which of the following graphs represents the velocity of a bouncing ball?




## *Synthesis problems and projects*

1. You and your friends Tom and Jerry are standing at the edge of a cliff with two identical rocks.

(a) Tom says "If I throw one of these rocks up and the other down, both with the same initial speed, they will hit the ground below the cliff at the same time, because in the relation $y = v_0 t + \frac{1}{2}at^2$ the three quantities $y$, $v_0$, and $a$ are the same for both cases." Do you agree? Explain.

(b) Jerry says: "Both rocks will have the same velocity just before hitting the ground." Do you agree? Explain.

2. A state trooper in a police car is parked on a highway. A car moving at 80 miles per hour (or 35.8 m/s) passes the police car. The trooper spends 5 s to call in a report and then accelerates to 100 miles per hour (or 44.7 m/s) to pursue the speeder.

(a) Draw graphs of $x(t)$ for both cars on the same coordinate system. Make the (not very realistic) assumption that the trooper changes his speed instantaneously. Let the origin of your graph ($x = 0$, $t = 0$) represent the place and time at which the speeding car passes the police car.

Use your graph to find the time when the trooper catches up with the speeder and the distance from the origin where this happens.

(b) Write down the mathematical statements for $x_1(t)$ for the speeding car. Develop a mathematical statement for $x_2(t)$ of the police car. (Check your expression by seeing where the police car is at $t = 5$ s.)

(c) From the two expressions answer the questions of part (a) analytically.

3. As you drive home with Leila she records the motion of your car by writing down the speedometer reading every second for 15 seconds, with the results shown in the following table.

| $t$ | $v$ |
|---|---|
| 1 | 17 |
| 2 | 17 |
| 3 | 17 |
| 4 | 17 |
| 5 | 16 |
| 6 | 15 |
| 7 | 14 |
| 8 | 13 |
| 9 | 12 |
| 10 | 11 |
| 11 | 10 |
| 12 | 9 |
| 13 | 8 |
| 14 | 7 |
| 15 | 6 |

(a) Plot the data for $v(t)$.

(b) Write two functions for $v(t)$. The first will be $v_1(t)$ for the time interval from $t = 0$ to $t = 4$ s.

The second will be $v_2(t)$ for $4\,\text{s} < t < 15\,\text{s}$. Let $(x = 0, t = 0)$ represent the position and time when she starts taking data. Use numbers for everything but $t$ and the units.

(c) Write the corresponding functions for $a_1(t), a_2(t), x_1(t)$, and $x_2(t)$.

(d) How far does the car move in the first four seconds?

(e) If the car continued with the same acceleration as between $t = 4\,\text{s}$ and $t = 15\,\text{s}$, at what values of $t$ and $x$ would it come to rest?

4. A tortoise and a snake start a race together, the tortoise moving at 0.30 m/s and the snake at 0.80 m/s. The snake stops after a minute to talk to a fellow snake for 3 min before starting to go again at the same speed as before. The tortoise wins the race by 2 m.

(a) Draw graphs of $x(t)$ for both on the same coordinates.

(b) Where are they after 1, 2, 3, 4, and 5 minutes?

(c) What are the length and the time of the race between the two?

(d) What is the equation for $x(t)$ of the tortoise? What is the equation of the line that represents the graph of $x(t)$ for the snake after her rest? (Check your expression by seeing whether it gives the answers for part [b] for 4 and 5 minutes.)

(e) Use the two equations to determine the answers to part (c) analytically.